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The Redheffer numbers and their applications

Ömür Deveci[1] and Anthony G. Shannon[2]

[1] Department of Mathematics, Faculty of Science and Letters, Kafkas University
36100 Kars, Turkey
e-mail: odeveci36@hotmail.com

[2] Fellow, Warrane College, The University of New South Wales
Kensington 2033, Australia
e-mail: tshannon38@gmail.com

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Abstract: In this paper, we define the Redheffer numbers and then we obtain their miscellaneous properties. Also, we study the Redheffer numbers modulo *m*. Furthermore, we define the Redheffer orbits and the basic Redheffer orbits of 2-generator and 3-generator groups, then we examine the lengths of the periods of these orbits. Finally, we obtain the Redheffer lengths and the basic Redheffer lengths of some special finite groups as applications of Redheffer orbits and the basic Redheffer orbits.

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1 Introduction and preliminaries

In [22], R. Redheffer defined the Redheffer matrix $R_n = \left[r_{ij} \right]_{n\times n}$ as follows:

$$r_{ij} = \begin{cases} 1 & \text{when } i \mid j \text{ or } j = 1, \\ 0 & \text{otherwise.} \end{cases} \quad \text{for } n > 3 \tag{1}$$

For example,

$$R_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In [22], it was obtained that

$$\det(R_n)=\sum_{k=1}^{n}\mu(k),$$

where μ is the classical Möbius functions.

The Redheffer matrix and its miscellaneous properties have been studied by some authors; see for example [2, 15, 26, 28].

Many of the numbers obtained by using linear recurrence relations and their miscellaneous properties have been studied by some authors; see for example, [4, 13, 16–18, 23–25, 29]. The study of recurrence sequences in groups began with the earlier work of Wall [27] where the ordinary Fibonacci sequences in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by several authors; see for example, [1, 3, 6–12, 19–21]. Now we extend the theory to the Redheffer sequences.

2 The Redheffer numbers

Now we define the Redheffer numbers by the following linear recurrence relations:

$$R(n)=\begin{cases}R(n-1)+R(n-2)+R(n-3), & n\equiv 1 \bmod 3,\\ R(n-3)+R(n-4), & n\equiv 2 \bmod 3, \quad \text{for } n>3\\ R(n-3)+R(n-5), & n\equiv 0 \bmod 3\end{cases} \tag{2}$$

where $R(1)=R(2)=0$ and $R(3)=1$.

For $n \geq 1$, by the inductive argument, we may write

$$R(3n+2)=R(3n+1)-R(3n),$$

$$R(3n+3)=R(3n+1)-R(3n-1),$$

and

$$R(3n)=R(3n-1)+1.$$

It is easy to show that

$$(R_3)^n=\begin{bmatrix}R(3n+3)+R(3n+2) & R(3n+1) & R(3n+1)\\ R(3n+1) & R(3n+3) & R(3n+2)\\ R(3n+1) & R(3n+2) & R(3n+3)\end{bmatrix} \tag{3}$$

for $n \geq 1$, which can be proved by mathematical induction. Since $\det R_3 = -1$, we can write the Simpson formula for the Redheffer numbers as:

$$\left(R(3n+3)-R(3n+2)\right)\left[\left(R(3n+3)+R(3n+2)\right)^2-2R(3n+1)^2\right]=(-1)^n.$$

By (2), we have

$$\begin{bmatrix}R(n+3)\\ R(n+2)\\ R(n+1)\\ R(n)\end{bmatrix}=\begin{bmatrix}0&0&1&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}R(n+2)\\ R(n+1)\\ R(n)\\ R(n-1)\end{bmatrix} \quad \text{for } n\equiv 2 \bmod 3,$$

$$\begin{bmatrix} R(n+5) \\ R(n+4) \\ R(n+3) \\ R(n+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R(n+4) \\ R(n+3) \\ R(n+2) \\ R(n+1) \end{bmatrix} \quad \text{for } n \equiv 1 \bmod 3$$

and

$$\begin{bmatrix} R(n+4) \\ R(n+3) \\ R(n+2) \\ R(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R(n+3) \\ R(n+2) \\ R(n+1) \\ R(n) \end{bmatrix} \quad \text{for } n \equiv 0 \bmod 3 \text{ such that } n \geq 3.$$

Then, for $n \geq 1$, we can write

$$\begin{bmatrix} R(3n+4) \\ R(3n+3) \\ R(3n+2) \\ R(3n+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R(3n+1) \\ R(3n) \\ R(3n-1) \\ R(3n-2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R(3n+1) \\ R(3n) \\ R(3n-1) \\ R(3n-2) \end{bmatrix}.$$

Theorem 2.1. Let n be a positive integer. Then

$$R(3n+1)^2 - R(3n+4)R(3n-2) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}.$$

Proof. We first note that

$$\begin{vmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1.$$

Also, by induction on n, we may write

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^n = \begin{cases} \begin{bmatrix} R(3n+4) & 0 & 0 & R(3n+1) \\ R(3n+3) & 0 & -1 & R(3n-1) \\ R(3n+3) & -1 & 0 & R(3n-1) \\ R(3n+1) & 0 & 0 & R(3n-2) \end{bmatrix} & \text{if } n \text{ is odd,} \\ \begin{bmatrix} R(3n+4) & 0 & 0 & R(3n+1) \\ R(3n+2) & 1 & 0 & R(3n) \\ R(3n+2) & 0 & 1 & R(3n) \\ R(3n+1) & 0 & 0 & R(3n-2) \end{bmatrix} & \text{if } n \text{ is even.} \end{cases}$$

So, the proof is easily seen. □

3 The Redheffer sequence modulo *m*

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence.

Reducing the Redheffer sequence by a modulus m, we can get a repeating sequence, denoted by

$$\{R^{(m)}(n)\} = \{R^{(m)}(1), R^{(m)}(2), R^{(m)}(3), \ldots, R^{(m)}(i), \ldots\}$$

where $R^{(m)}(i) = R(i) \bmod m$. They have the same recurrence relation as in (2).

Theorem 3.1. $\{R^{(m)}(n)\}$ is a simply periodic sequence.

Proof. Let $A = \{(a_1, a_2, a_3) \mid 0 \le a_i \le m-1\}$. Then we have $|A| = m^3$. Thus, it is easy to see that the sequence repeats since there are only a finite number m^3 of triples of terms possible, and the recurrence of a triple results in recurrence of all following terms. So if $R^{(m)}(i+3) \equiv R^{(m)}(j+3)$, $R^{(m)}(i+2) \equiv R^{(m)}(j+2)$, $R^{(m)}(i+1) \equiv R^{(m)}(j+1)$ and $i > j$, then $i \equiv j \bmod 3$. From the definition of the Redheffer sequence we can easily get that $R^{(m)}(i-1) \equiv R^{(m)}(j-1)$, $R^{(m)}(i-2) \equiv R^{(m)}(j-2)$, …, $R^{(m)}(i-j+1) \equiv R^{(m)}(1)$, which implies that the $\{R^{(m)}(n)\}$ is a simply periodic sequence. □

Let $l_r(m)$ denote the smallest periods of $\{R^{(m)}(n)\}$.

For a given matrix $M = [m_{ij}]$ with m_{ij}'s being integers, $M \pmod m$ means that all entries of M are reduced modulo m, that is, $M \pmod m = (m_{ij} \pmod m)$.

Let $\langle R_3 \rangle_m = \{(R_3)^i \pmod m \mid i \ge 0\}$. Since $\det(R_3)^i = (-1)^i$, $\langle R_3 \rangle_m$ is the cyclic group, we denote the order of the group $\langle R_3 \rangle_m$ by $|\langle R_3 \rangle_m|$.

Let p be a prime. Then it is easy to see from (3) that $l_r(p) = 3 \cdot |\langle R_3 \rangle_p|$.

Example. Since

$$\{R^{(3)}(n)\} = \{0,0,1,1,0,1,2,1,2,2,0,1,0,2,0,2,2,0,1,1,2,1,2,0,0,0,1,1,0,1,\ldots\}$$

and

$$(R_3)^8 \equiv I \bmod 3,$$

we have $l_r(3) = 3 \cdot |\langle R_3 \rangle_3| = 24$.

Theorem 3.2. Let u be the largest positive integer and let p be a prime such that $l_r(p) = l_r(p^u)$. Then $l_r(p^v) = p^{v-u}.l_r(p)$ for every $v \ge u$.

Proof. Let a be a positive integer let $(R_3)^{l_r(p^{a+1})} \equiv I \pmod{p^{a+1}}$, then $(R_3)^{l_r(p^{a+1})} \equiv I \pmod{p^a}$ where I is a 3×3 identity matrix. Thus we obtain that $l_r(p^a)$ divides $l_r(p^{a+1})$. Also, writing $(R_3)^{l_r(p^a)} = I + (r_{ij}^{(a)} \cdot p^a)$, by the binomial theorem, we obtain

$$(R_3)^{l_r(p^a) \cdot p} = \left(I + \left(r_{ij}^{(a)} \cdot p^a\right)\right)^p = \sum_{i=0}^{p} \binom{p}{i} \left(r_{ij}^{(a)} \cdot p^a\right)^i \equiv I \left(\bmod\ p^{a+1}\right),$$

which yields that $l_r(p^{a+1})$ divides $l(p^a).p$. Thus, $l_r(p^{a+1}) = l_r(p^a)$ or $l_r(p^{a+1}) = l_r(p^a).p$, and the latter holds if and only if there is $r_{ij}^{(a)}$, which is not divisible by p.

Since u is the largest positive integer such that $l_r(p) = l_r(p^u)$, $l_r(p^u) \neq l_r(p^{u+1})$, there is a $r_{ij}^{(u+1)}$, which is not divisible by p. So we get that $l_r(p^{u+1}) \neq l_r(p^{u+2})$.

The proof is completed by induction on u. □

Theorem 3.3. If

$$m = \prod_{i=1}^{k} p_i^{e_i}, (k \geq 1),$$

where p_i's are distinct primes, then $l_r(m) = \text{lcm}\left[l_r\left(p_1^{e_1}\right), l_r\left(p_2^{e_2}\right), \cdots, l_r\left(p_k^{e_k}\right)\right]$.

Proof. Since $l_r(p_i^{e_i})$ is the length of the period of the sequence $\{R^{(p_i^{e_i})}(n)\}$, this sequence repeats only after blocks of length $\lambda \cdot l_r(p_i^{e_i})$, $(\lambda \in \mathbb{N})$. Since also $l_r(m)$ is the length of the period $\{R^{(m)}(n)\}$, the sequence $\{R^{(p_i^{e_i})}(n)\}$ repeats after $l_r(m)$ terms for all values i. Then $l_r(m)$ is the form $\lambda \cdot l_r(p_i^{e_i})$ for all values of i, and since any such number gives a period of $\{R^{(m)}(n)\}$. Thus it is verified that $l_r(m) = \text{lcm}\left[l_r\left(p_1^{e_1}\right), l_r\left(p_2^{e_2}\right), \cdots, l_r\left(p_k^{e_k}\right)\right]$. □

4 The Redheffer sequence and the basic Redheffer sequence in groups

Let G be a finite j-generator group and let X be the subset of

$$\underbrace{G \times G \times G \ldots \times G}_{j},$$

such that $(x_1, x_2, \ldots, x_j) \in X$, if and only if G is generated by $x_1, x_2, \ldots, x_j$. We call $(x_1, x_2, \ldots, x_j)$ a generating j-tuple for G.

Each generating *j-tuple* $(x_0, x_1, \ldots, x_{j-1}) \in X$ maps to $|\text{Aut}\, G|$ distinct elements of X under the action of elements of Aut G. Hence there are

$$d_j(G) = |X| \,/\, |\text{Aut}\, G|$$

non-isomorphic generating *j-tuples* for G, where $|X|$ means the number of elements of X [7].

The notation $d_j(G)$ was introduced in [14].

Definition 4.1. Let G be a finitely generated group $G = \langle A \rangle$, where $A = \{x_1, x_2\}$ or $A = \{x_1, x_2, x_3\}$. The Redheffer orbit of the group G is defined as follows:

i. Let G be a 2-generator group. For a generating pair $(x_1, x_2) \in X$, the Redheffer orbit $R(G)_{\{x_1,x_2\}}$ is defined by the sequence $\{a_i\}$ of elements of G such that

$$a_1 = x_1, a_2 = x_2, a_3 = e,$$

$$a_{i+3} = \begin{cases} a_i a_{i+1} a_{i+2}, & i \equiv 1 \bmod 3, \\ a_{i-1} a_i, & i \equiv 2 \bmod 3, \quad \text{for } i \geq 1. \\ a_{i-2} a_i, & i \equiv 0 \bmod 3 \end{cases}$$

ii. Let G be a 3-generator group. For a generating triple $(x_1, x_2, x_3) \in X$, the Redheffer orbit $R(G)_{\{x_1,x_2,x_3\}}$ is defined by the sequence $\{a_i\}$ of elements of G such that

$$a_1 = x_1,\ a_2 = x_2,\ a_3 = x_3,$$

$$a_{i+3} = \begin{cases} a_i a_{i+1} a_{i+2}, & i \equiv 1 \bmod 3, \\ a_{i-1} a_i, & i \equiv 2 \bmod 3, \quad \text{for } i \ge 1. \\ a_{i-2} a_i, & i \equiv 0 \bmod 3 \end{cases}$$

Theorem 4.1. The Redheffer orbit of a finite group is simply periodic.

Proof. Let n be the order of G. Since there are n^3 distinct 3-tuples of elements of G, at least one of the 3-tuples appears twice in a Redheffer orbit of G. Thus, the subsequence following this 3-tuple repeats. Because of the repeating, the Redheffer orbit is periodic.

Since the Redheffer orbit periodic, there exist natural numbers i and j, with $i > j$, such that

$$a_{i+1} = a_{j+1},\ a_{i+2} = a_{j+2},\ a_{i+3} = a_{j+3}.$$

By the defining relation of the Redheffer orbit, we know that

$$\begin{aligned} & a_n (a_{n-1})^{-1} (a_{n-2})^{-1} = a_{n-3} && \text{if } n \equiv 1 \bmod 3, \\ & a_n (a_{n-3})^{-1} = a_{n-4} && \text{if } n \equiv 2 \bmod 3, \\ & a_n (a_{n-3})^{-1} = a_{n-5} && \text{if } n \equiv 0 \bmod 3, \end{aligned}$$

for $n > 3$. Therefore, $a_i = a_j$, and it then follows that

$$a_{i-1} = a_{j-1},\ a_{i-2} = a_{j-2}, \cdots,\ a_{j-(j-1)} = a_{i-(j-1)} = a_1,$$

which implies that the sequence is simply periodic.

We denote the length of the period of the Redheffer orbit $R(G)_A$ $LR(G)_A$ and we call the Redheffer length of G with respect to the generating set A.

Definition 4.2. Let G be a finite group. If every element of G appears in the Redheffer orbit of G, then the group G is called Redheffer sequenceable.

Now Aut G consists of all isomorphisms $\theta : G \to G$ and if $\theta \in$ Aut G and $(x_1, x_2, \ldots, x_j) \in X$, then $(x_1\theta, x_2\theta, \ldots, x_j\theta) \in X$.

For a subset $B \subseteq G$ and $\theta \in$ Aut G the image of B under θ is $B\theta = \{b\theta : b \in B\}$.

Lemma 4.1. Let $\theta \in$ Aut G. Then $(R(G)_A)\theta = R(G)_{A\theta}$.

Proof. Let $R(G)_A = \{a_i\}$ Since

$$a_{i+3}\theta = \begin{cases} a_i\theta a_{i+1}\theta a_{i+2}\theta, & i \equiv 1 \bmod 3, \\ a_{i-1}\theta a_i\theta, & i \equiv 2 \bmod 3, \quad \text{for } i \ge 1 \\ a_{i-2}\theta a_i\theta, & i \equiv 0 \bmod 3 \end{cases}$$

$$\{a_i\}\theta = \{a_i\theta\}.$$

Thus, the result is obvious. □

Suppose that k elements of Aut G map $R(G)_A$ into itself. Then there are |Aut G| / k distinct Redheffer orbits $R(G)_A$ for $\theta \in$ Aut G.

Definition 4.3. Let G be a finitely generated group $G = \langle A \rangle$, where $A = \{x_1, x_2\}$ or $A = \{x_1, x_2, x_3\}$. The Redheffer basic orbit of the group G is defined as follows:

i. Let G be a 2-generator group. For a generating pair $(x_1, x_2) \in X$ we define the basic Redheffer orbit $\overline{R(G)_{\{x_1,x_2,x_3\}}}$ of the basic length m to be the sequence $\{b_i\}$ of elements of G such that $b_1 = x_1$, $b_2 = x_2$, $b_3 = e$, each element is defined by

$$b_{i+3} = \begin{cases} b_i b_{i+1} b_{i+2}, & i \equiv 1 \bmod 3, \\ b_{i-1} b_i, & i \equiv 2 \bmod 3, \quad \text{for } i \geq 1 \\ b_{i-2} b_i, & i \equiv 0 \bmod 3 \end{cases}$$

where $m \geq 1$ is the least integer with

$$b_1 = b_{m+1}\theta, b_2 = b_{m+2}\theta, \cdots, b_6 = b_{m+6}\theta,$$

for some $\theta \in \operatorname{Aut} G$.

ii. Let G be a 3-generator group. For a generating pair $(x_1, x_2, x_3) \in X$ we define the basic Redheffer orbit $\overline{R(G)_{\{x_1,x_2,x_3\}}}$ of the basic length m to be the sequence $\{b_i\}$ of elements of G such that $b_1 = x_1$, $b_2 = x_2$, $b_3 = x_3$, each element is defined by

$$b_{i+3} = \begin{cases} b_i b_{i+1} b_{i+2}, & i \equiv 1 \bmod 3, \\ b_{i-1} b_i, & i \equiv 2 \bmod 3, \quad \text{for } i \geq 1 \\ b_{i-2} b_i, & i \equiv 0 \bmod 3 \end{cases}$$

where $m \geq 1$ is the least integer with

$$b_1 = b_{m+1}\theta, b_2 = b_{m+2}\theta, \cdots, b_6 = b_{m+6}\theta,$$

for some $\theta \in \operatorname{Aut} G$.

We denote the length of the period of the basic Redheffer orbit $\overline{R(G)_A}$ by $BLR(G)_A$ and we call the basic Redheffer length of G with respect to the generating set A.

It is obvious that the basic Redheffer orbit $\overline{R(G)_A}$ is finite containing m element. If G is a 2-generator group, we have an integer i such that $1 \leq i \leq 5$ and b_{m+i}, b_{m+i+1} generate G. Let G be a 3-generator group, then it is easy to see that b_{m+i}, b_{m+i+1}, b_{m+i+3} generate G. Thus it is verified that θ is uniquely determined.

Theorem 4.2. Let G be a finite group and let $G = \langle A \rangle$ such that $A = \{x_1, x_2\}$ or $A = \{x_1, x_2, x_3\}$. If $LR(G)_A = n$ and $BLR(G)_A = m$, then m divides n and there are n/m elements of Aut G, which map $R(G)_A$ into itself.

Proof. We have $n = mt$ where t is order of automorphism $\theta \in \operatorname{Aut} G$ since

$$R(G)_A = \overline{R(G)_A} \cup \overline{R(G)_{A\theta}} \cup \overline{R(G)_{A\theta^2}} \cup \cdots$$

and $BLR(G)_A = BLR(G)_{A\theta}$. So we get that $1, \theta, \theta^2, \cdots, \theta^{t-1}$ map $R(G)_A$ into itself.

From the definitions it is clear that the Redheffer length and the basic Redheffer length of a group depend on the chosen generating set and the order initial elements of the orbits. □

5 Applications

Theorem 5.1. $LR(S_3)_{\{x,y\}} = 3 \cdot BLR(S_3)_{\{x,y\}} = 36$.

Proof. We first note that the symmetric group S_3 of order 6 is presented by

$$\langle x, y : x^2 = y^3 = (xy)^2 = e \rangle.$$

The Redheffer orbit $R(S_3)_{\{x,y\}}$ is

$$x, y, e, xy, xy, x, x, e, y^2, yx, x, yx, xy, y, e, yx, yx, xy, xy,$$
$$e, y^2, x, xy, x, yx, y, e, x, x, yx, yx, e, y^2, xy, yx, xy, x, y, e, \cdots.$$

Thus we obtain $LR(S_3)_{\{x,y\}} = 36$ and $BLR(S_3)_{\{x,y\}} = 12$ since $x\theta = yx$ and $y\theta = y$ where θ is the inner automorphism induced by conjugation by y^2. Also we have that the symmetric group S_3 is Redheffer sequenceable. □

Theorem 5.2. $LR(Q_8)_{\{x,y\}} = 2 \cdot BLR(Q_8)_{\{x,y\}} = 24$.

Proof. We first note that the quarternion group Q_8 of order 8 is presented by

$$\langle x, y : x^4 = e,\ y^2 = x^2,\ yx = x^3 y \rangle.$$

The Redheffer orbit $R(Q_8)_{\{x,y\}}$ is

$$x, y, e, xy, xy, x, x^3, x^2, y, xy, x, yx, x^3, y, e,$$
$$yx, yx, x^3, x, x^2, y, yx, x^3, xy, x, y, e \cdots.$$

Thus we obtain $LR(Q_8)_{\{x,y\}} = 24$ and $BLR(Q_8)_{\{x,y\}} = 12$ since $x\theta = x^3$ and $y\theta = y$ where θ is the inner automorphism induced by conjugation by y. Also we have that the group Q_8 is not a Redheffer sequenceable group since y^3 is an element of Q_8 but y^3 does not appear in the Redheffer orbit $R(Q_8)_{\{x,y\}}$. □

Theorem 5.3. i. $LR(D_4)_{\{x,y\}} = BLR(D_4)_{\{x,y\}} = 12$.

ii. $LR(D_{2n})_{\{x,y\}} = 2 \cdot BLR(D_{2n})_{\{x,y\}} = 24$ for $n \geq 3$.

Proof. We first note that the dihedral group D_{2n} of order $2n$ is presented by

$$\langle x, y : x^2 = y^2 = (xy)^n = e \rangle.$$

The Redheffer orbit $R(D_{2n})_{\{x,y\}}$ is

$$x, y, e, xy, xy, x, (xy)^2 x, (xy)^2, xyx, yx, x, xy, yxy, y, e,$$
$$yx, yx, yxy, (yx)^3 y, (yx)^2, (yx)^2 y, xy, yxy, yx, x, y, e, \cdots.$$

i. $LR(D_4)_{\{x,y\}} = BLR(D_4)_{\{x,y\}} = 12$ since $x\theta = x$ and $y\theta = y$ where θ is the identity transform.

ii. If $n \geq 3$, $LR(D_{2n})_{\{x,y\}} = 24$ and $BLR(D_{2n})_{\{x,y\}} = 12$ since $x\theta = yxy$ and $y\theta = y$ where θ is the inner automorphism induced by conjugation by y.

It is easy to see that the groups D_4 (four-group), D_6, D_8 and D_{10} are Redheffer sequenceable. Also we have that for $n \geq 6$ the group D_{2n} is not a Redheffer sequenceable group since $(xy)^3$ is an element of D_{2n} for $n \geq 6$ but $(xy)^3$ does not appear in the Redheffer orbit $R(D_{2n})_{\{x,y\}}$. □

Theorem 5.4. $LR([2,q])_{\{x,y,z\}} = 6q$ and

$$BLR([2,q])_{\{x,y,z\}} = \begin{cases} \frac{3q}{2}, & n \text{ is even,} \\ 3q, & n \text{ is odd.} \end{cases}$$

Proof. We first note that the group $[2, q]$ of order $4q$ is presented by

$$\left\langle x, y, z : x^2 = y^2 = z^2 = (xy)^2 = (xz)^2 = (yz)^q = e \right\rangle.$$

It is clear that $[2,q] \cong C_2 \times D_{2q}$ where C_2 is a cyclic group of order 2.

For more information on the groups $[p, q]$ see [5, pp. 35–38].

The Redheffer orbit $R([2, q])_{\{x,y,z\}}$ is

$$x, y, z, xyz, xy, xz, x(yz)^2, yzy, y, x(yz)^3, x(yz)^3 y, x(yz)^2 y,$$
$$x(yz)^4, (yz)^6 y, (yz)^5 y, x(yz)^5, x(yz)^{10} y, x(yz)^9 y, x(yz)^6,$$
$$(yz)^{15} y, (yz)^{14} y, x(yz)^7, x(yz)^{21} y, x(yz)^{20} y, x(yz)^8, (yz)^{28} y, (yz)^{27} y, \cdots.$$

Using the above, the sequence becomes:

$$x, y, z, xyz, xy, xz, \cdots,$$
$$x_{6i+1} = x(yz)^{2i}, x_{6i+2} = (yz)^{2i^2-i} y, x_{6i+3} = (yz)^{2i^2-i-1} y,$$
$$x_{6i+4} = x(yz)^{2i+1}, x_{6i+5} = x(yz)^{2i^2+i} y, x_{6i+6} = x(yz)^{2i^2+i-1} y, \cdots.$$

So, we need the smallest $i \in N$ such that $i = qv$ for $v \in N$. If we choose $i = q$, then we obtain $LR([2, q])_{\{x,y,z\}} = 6q$.

If $q \equiv 0 \bmod 4$, then

$$BLR([2,q])_{\{x,y,z\}} = \frac{3q}{2},$$

since $x\theta = x(yz)^{\frac{q}{2}}$, $y\theta = (yz)^{a_1} y$ and $z\theta = (yz)^{a_1 - 1} y$ where $\frac{9q^2}{8} - \frac{3q}{4} \equiv a_1 \bmod q$ and θ is an outer automorphism of order 4.

If $q \equiv 2 \bmod 4$, then

$$BLR([2,q])_{\{x,y,z\}} = \frac{3q}{2}$$

since $x\theta = x(yz)^{\frac{q}{2}}$, $y\theta = x(yz)^{a_2} y$ and $z\theta = x(yz)^{a_2 - 1} y$ where

$$\frac{9(q-2)^2}{8} + \frac{15(q-2)}{4} + 3 \equiv a_2 \bmod q$$

and θ is an outer automorphism of order 4.

If n is odd, then $BLR([2,q])_{\{x,y,z\}} = 3q$ since $x\theta = x$, $y\theta = xy$ and $z\theta = xz$ where θ is an outer automorphism of order 2.

Also we have that the group $[2, q]$ is not a Redheffer sequenceable group, since yz is an element of $[2, q]$ but yz does not appear in the Redheffer orbit $R([2,q])_{\{x,y,z\}}$. □

Theorem 5.5. i. $LR(A_4)_{\{x,y\}} = 3 \cdot BLR(A_4)_{\{x,y\}} = 18$.

ii. $LR(A_4)_{\{x,y\}} = 3 \cdot BLR(A_4)_{\{x,y\}} = 72$.

Proof. We first note that the alternating group A_4 of order 12 is presented by

$$\left\langle x, y, z : x^2 = y^3 = z^3 = xyz = e \right\rangle$$

or

$$\left\langle x, y : x^2 = y^3 = (xy)^3 = e \right\rangle.$$

Firstly, let us consider the 3-generator case. It is clear that $z = y^2x$.

The orbit $R(A_4)_{\{x,y,z\}}$ is

$$x, y, z, e, xy, yxy, yxy^2, xy, yxy, e, xyx, y^2, y^2xy, xyx, y^2, e, y, y^2x, x, y, z, \cdots.$$

Thus we obtain $LR(A_4)_{\{x,y,z\}} = 18$ and $BLR(A_4)_{\{x,y,z\}} = 6$, since $x\theta = y^2xy$, $y\theta = xyx$ and $z\theta = y^2$, where θ is the inner automorphism induced by conjugation by xy^2. Also we have that the alternating group A_4 for generating set $\{x, y, z\}$ is not a Redheffer sequenceable group since ten elements of the group A_4 appear in the orbit $R(A_4)_{\{x,y,z\}}$ but the order of the group A_4 is twelve.

Secondly, let us consider the 2-generator case.

The orbit $R(A_4)_{\{x,y\}}$ is

$$\begin{aligned}
&x, y, e, xy, xy, x, y^2, y^2x, xyx, y^2x, yx, yxy^2, e, yxy^2, yxy, y^2, yxy^2, yxy, y,\\
&xy^2, xy, xyx, yxy^2, yxy, y^2xy, y, e, xyx, xyx, y^2xy, y^2, yxy, yx, yxy, xy, x, e,\\
&x, xy^2, y^2, x, xy^2, y, y^2x, xyx, yx, x, xy^2, yxy^2, y, e, yx, yx, yxy^2, y^2, xy^2, xy, xy^2,\\
&xyx, y^2xy, e, y^2xy, y^2x, y^2, y^2xy, y^2x, y, yxy, yx, xy, y^2xy, y^2x, x, y, e, \cdots,
\end{aligned}$$

which has period 72, that is, $LR(A_4)_{\{x,y\}} = 72$, $BLR(A_4)_{\{x,y\}} = 24$, since $x\theta = yxy^2$ and $y\theta = y$, where θ is the inner automorphism induced by conjugation by y. Also we have that the alternating group A_4 for generating set $\{x, y\}$ is Redheffer sequenceable. □

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