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An identical equation for arithmetic functions of several variables and applications

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Abstract: An identical equation for arithmetic functions is proved generalizing the 2-variable case due to Venkataraman. It is then applied to characterize multiplicative functions which are variable-separated, and to deduce interesting properties of generalized Ramanujan sums.
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1 Introduction

For $r \in \mathbb{N}$, let $\mathcal{A}_r := \{F : \mathbb{N}^r \to \mathbb{C}\}$ be the set of complex-valued arithmetic functions in r variables. For $F, G \in \mathcal{A}_r$, their addition and Dirichlet convolution are defined, respectively, by

$$(F+G)(n_1, \ldots, n_r) = F(n_1, \ldots, n_r) + G(n_1, \ldots, n_r)$$
$$(F*G)(n_1, \ldots, n_r) = \sum_{d_1 \mid n_1} \cdots \sum_{d_r \mid n_r} F(d_1, \ldots, d_r) G(n_1/d_1, \ldots, n_r/d_r).$$

For a general reference, see also [3]. For $F \in \mathcal{A}_r$, define its i-th $(i = 1, \ldots, r)$ **component function** $F_i \in \mathcal{A}_1$ to be

$$F_i(n) = F(1, \ldots, 1, n, 1, \ldots, 1),$$

where the right-hand n appears at the i-th coordinate. Denote by $\mathbf{1} \in \mathcal{A}_r$ the **constant** 1 **function**, i.e., $\mathbf{1}(n_1, \dots, n_r) \equiv 1$ and by $\mathbf{0} \in \mathcal{A}_r$ the **constant** 0 **function**, i.e., $\mathbf{0}(n_1, \dots, n_r) \equiv 0$.
A function $F \in \mathcal{A}_r \setminus \{0\}$ is said to be **multiplicative** if

$$F(m_1 n_1, \dots, m_r n_r) = F(m_1, \dots, m_r) F(n_1, \dots, n_r)$$

holds for any $m_1, \dots, m_r, n_1, \dots, n_r \in \mathbb{N}$ such that $\gcd(m_1 \cdots m_r, n_1 \cdots n_r) = 1$, and let $\mathcal{M}_r := \{F \in \mathcal{A}_r\ ;\ F \text{ is multiplicative}\}$. Clearly, if $F \in \mathcal{M}_r$, then $F(1, \dots, 1) = 1$. A function $F \in \mathcal{A}_r \setminus \{0\}$ is said to be **firmly multiplicative** if

$$F(m_1 n_1, \dots, m_r n_r) = F(m_1, \dots, m_r) F(n_1, \dots, n_r)$$

holds for any $m_1, \dots, m_r, n_1, \dots, n_r \in \mathbb{N}$ such that $\gcd(m_1, , n_1) = \cdots = \gcd(m_r, n_r) = 1$, and let $\mathcal{F}_r := \{F \in \mathcal{A}_r\ ;\ F \text{ is firmly multiplicative}\}$. A function $F \in \mathcal{A}_r \setminus \{0\}$ is said to be **completely multiplicative** if

$$F(m_1 n_1, \dots, m_r n_r) = F(m_1, \dots, m_r) F(n_1, \dots, n_r)$$

holds for any $m_1, \dots, m_r, n_1, \dots, n_r \in \mathbb{N}$, and let

$$\mathcal{C}_r := \{F \in \mathcal{A}_r\ ;\ F \text{ is completely multiplicative}\}.$$

Clearly, $\mathcal{C}_r \subsetneq \mathcal{F}_r \subsetneq \mathcal{M}_r$. Note that there is no universal agreement regarding multiplicative functions, e.g., in [1], the term 'mutiplicative' is used for 'firmly multiplicative'.

In the 1-variable case, the Souriau–Hsu–Möbius function ([2,6]) is defined, for $\alpha \in \mathbb{C}$, by

$$\mu_\alpha(n) = \prod_{p|n} \binom{\alpha}{\nu_p(n)} (-1)^{\nu_p(n)},$$

where $n = \prod p^{\nu_p(n)}$ denotes the unique prime factorization of $n \in \mathbb{N}$, $\nu_p(n)$ being the largest exponent of the prime p that divides n; when $\alpha = 1$, this corresponds to the classical Möbius function $\mu_1 := \mu$. The r-variable Souriau–Hsu–Möbius function is defined by

$$M_\alpha(n_1, \dots, n_r) = \mu_\alpha(n_1) \cdots \mu_\alpha(n_r).$$

The following basic results are easily checked, cf. [7].

I. The set $(\mathcal{A}_r, +, *)$ is commutative ring with identity I (with respect to $*$), where

$$I(n_1 \dots, n_r) = \begin{cases} 1 & \text{if } n_1 = \cdots = n_r = 1, \\ 0 & \text{if } n_1 \cdots n_r > 1; \end{cases}$$

note that $I(n_1 \dots, n_r) = \delta(n_1) \cdots \delta(n_r)$, where $\delta \in \mathcal{A}_1$ is the 1-dimensional identity function with respect to the Dirichlet convolution, i.e., $\delta(n) = 1$ if $n = 1$, and $\delta(n) = 0$ if $n > 1$;

II. A function $F \in \mathcal{A}_r$ has a uniquely determined inverse with respect to the convolution, denoted by F^{-1}, if and only if $F(1, \dots, 1) \neq 0$;

III. The inverse of $\mathbf{1}$ is the Möbius function $M_1(n_1, \dots, n_r) = \mu(n_1) \cdots \mu(n_r)$.

2 Identitical equation and applications

The following basic results are easily derived ([7, Propositions 1, 2]).

Proposition 2.1. **I.** *A function $F \in \mathcal{A}_r$ is firmly multiplicative if and only if there exist multiplicative functions $f_1, \ldots, f_r \in \mathcal{M}_1$ (each of a single variable) such that*

$$F(n_1, \ldots, n_r) = f_1(n_1) \cdots f_r(n_r) \quad (n_1, \ldots, n_r \in \mathbb{N}).$$

In this case, f_i $(i = 1, \ldots, r)$ is the i-th component function of F, i.e., $f_i(n) = F_i(n)$.

II. *A function $F \in \mathcal{A}_r$ is completely multiplicative if and only if there exist completely multiplicative functions $f_1, \ldots, f_r \in \mathcal{C}_1$ (each of a single variable) such that*

$$F(n_1, \ldots, n_r) = f_1(n_1) \cdots f_r(n_r) \quad (n_1, \ldots, n_r \in \mathbb{N}).$$

In this case, f_i $(i = 1, \ldots, r)$ is the i-th component function of F, i.e., $f_i(n) = F_i(n)$.

Proposition 2.1 suggests the following notion. Let $F \in \mathcal{A}_r$. We say that F is variable-separated, denoted by $F \in V_{sep}$, if F can be written as

$$F(n_1, \ldots, n_r) = f_1(n_1) f_2(n_2) \cdots f_r(n_r), \tag{2.1}$$

where $f_i \in \mathcal{A}_1$ and $f_i(1) = f_j(1) \in \mathbb{R} \setminus \{0\} \quad (1 \leq i, j \leq r)$. For $F \in V_{sep}$ as given in (2.1), if $F \in \mathcal{M}_r$, then $f_1(1) = \cdots = f_r(1) = 1$. Note also that the concept of variable-separated function is trivial in the 1-variable case. Variable-separated functions are plentiful, because most well-known classical functions such as the 1-constant function $\mathbf{1}$, the generalized Möbius function M_α, the convolution identity function I and all firmly multilicative functions ([7]), are variable-separated. Yet, there are indeed arithmetic functions which are not variable-separated as seen from the example of the 2-variable function $f(n_1, n_2) := \gcd(n_1, n_2) \in \mathcal{A}_2$, which is multiplicative, but not variable-separated.

Proposition 2.1 indicates that firmly and completely multiplicative functions belong to V_{sep}, and it is mentioned in [7, p. 5] that *there is no similar characterization for multiplicative functions. This is only partially true as we shall give a characterization based on the notion of cardinal function, which generalizes the 2-variable case of Venkataraman [8, Theorem 3.1.2].* To do so, we need an r-variable version of the so-called **identical equation**. In the 1-variable case, it is well-known that a function $f \in \mathcal{M}_1$ satisfies an identity called its identical equation of the form

$$f(nr) = \sum_{t|n} \sum_{d|r} f(n/t) f(r/d) f^{-1}(td) \psi(t, d),$$

where $\psi(n, r) = \begin{cases} (-1)^v & \text{if } n \text{ and } r \text{ contain the same distinct } v \text{ prime factors,} \\ 0 & \text{otherwise.} \end{cases}$

This result is generalized to the 2-variable case by Venkataraman [8, p. 522], see also [5, Chapter VII]. Before stating Venkatarama's result, we need one more definition. A function $F(n_1, \ldots, n_r) \in \mathcal{M}_r$ is called **cardinal function** if all its component functions are equal to the 1-dimensional identity function, i.e., $F_i(n) = \delta(n)$ $(i = 1, \ldots, r)$. The identical equation of Venkataraman ([8, Theorem 3.1.2]) states that: *every multiplicative function* $f(n, r) \in \mathcal{M}_2$ *satisfies the identity*

$$f(n, r) = \sum_{d_1|n_1} \sum_{d_2|n_2} f(n_1/d_1, 1) f(1, n_2/d_2) K(d_1, d_2),$$

where $K(n_1, n_2) = \sum_{d_1|n_1} \sum_{d_2|n_2} f^{-1}(n_1/d_1, 1) f^{-1}(1, n_2/d_2) f(d_1, d_2)$ *is a cardinal function.* Our generalization of Venkataraman's result is:

Theorem 2.2. *If* $F(n_1, n_2, \ldots, n_r) \in \mathcal{M}_r$*, then*

$$\begin{aligned} F(n_1, n_2, \ldots, n_r) &= (F_1 F_2 \cdots F_r) * K\ (n_1, n_2, \ldots, n_r) \\ &= \sum_{d_1|n_1} \sum_{d_2|n_2} \cdots \sum_{d_r|n_r} F_1(n_1/d_1) F_2(n_2/d_2) \cdots F_r(n_r/d_r) K_F(d_1, d_2, \ldots, d_r), \end{aligned}$$

where $F_i \in \mathcal{M}_1$ $(1 \leq i \leq r)$ *is the* i*-th component function of* F*, and* K_F *is a cardinal function, called the* **cardinal component** *of* F*, defined by*

$$\begin{aligned} K_F(n_1, n_2, \ldots, n_r) &= (F_1^{-1} F_2^{-1} \cdots F_r^{-1}) * F\ (n_1, n_2, \ldots, n_r) \\ &= \sum_{d_1|n_1} \sum_{d_2|n_2} \cdots \sum_{d_r|n_r} F_1^{-1}(n_1/d_1) F_2^{-1}(n_2/d_2) \cdots F_r^{-1}(n_r/d_r) F(d_1, d_2, \ldots, d_r). \end{aligned}$$

Proof. Since $f \in \mathcal{M}_r$, all its component functions are also multiplicative, i.e., $F_j \in \mathcal{M}_1$, and so is the function $g(n_1, n_2, \ldots, n_r) := F_1(n_1) F_2(n_2) \cdots F_r(n_r) \in \mathcal{M}_r$, considered as a function in r variables $(n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$. It is easily checked that

$$g^{-1}(n_1, n_2, \ldots, n_r) = F_1^{-1}(n_1) F_2^{-1}(n_2) \cdots F_r^{-1}(n_r).$$

Thus,

$$\begin{aligned} K_F(n_1, n_2, \ldots, n_r) &= \sum_{d_1|n_1} \sum_{d_2|n_2} \cdots \sum_{d_r|n_r} g^{-1}(n_1/d_1, n_2/d_2, \ldots, n_r/d_r) F(d_1, d_2, \ldots, d_r) \\ &= (g^{-1} * F)(n_1, n_2, \ldots, n_r) \end{aligned}$$

is a cardinal function with $F(n_1, n_2, \ldots, n_r) = (g * K)(n_1, n_2, \ldots, n_r)$, as desired. □

We now return to the problem of characterizing multiplicative functions in V_{sep}.

Theorem 2.3. *Let* $F \in \mathcal{M}_r$*. Then* $F \in V_{sep}$ *if and only if* $K_F(n_1, \ldots, n_r)$*, the cardinal component of* F*, is equal to* $\delta(n_1)\delta(n_2) \cdots \delta(n_r)$*.*

Proof. If the cardinal component of F is $\delta(n_1)\delta(n_2) \cdots \delta(n_r)$, using Venkataraman's identity (Theorem 2.2), we have

$$\begin{aligned} F(n_1, n_2, \ldots, n_r) &= \sum_{d_1|n_1} \sum_{d_2|n_2} \cdots \sum_{d_r|n_r} F_1(n_1/d_1) F_2(n_2/d_2) \cdots F_r(n_r/d_r) \delta(d_1)\delta(d_2) \cdots \delta(d_r) \\ &= (F_1 * \delta)(n_1)(F_2 * \delta)(n_2) \cdots (F_r * \delta)(n_r) = F_1(n_1) F_2(n_2) \cdots F_r(n_r) \in V_{sep}. \end{aligned}$$

Assume next that $F(n_1, n_2, \ldots, n_r) = f_1(n_1)f_2(n_2)\cdots f_r(n_r) \in V_{sep}$. Since $F \in \mathcal{M}_r$, it is easily checked that $f_i \in \mathcal{M}_1$ $(1 \le i \le r)$, and so $f_i(1) = 1$ $(1 \le i \le r)$. Now the component functions of F satisfy $F_i(n) = f_1(1)\cdots f_{i-1}(1)f_i(n)f_{i+1}(1)\cdots f_r(1) = f_i(n)$ $(1 \le i \le r)$, so that f_i is simply the i-th component function of F. Thus, the cardinal component of f becomes

$$\begin{aligned} K(n_1, n_2, \ldots, n_r) &= \sum_{d_1|n_1}\cdots\sum_{d_r|n_r} F_1^{-1}(n_1/d_1)\cdots F_r^{-1}(n_r/d_r)F_1(d_1)\cdots F_r(d_r) \\ &= (F_1^{-1} * F_1)(n_1)\cdots(F_r^{-1} * F_r)(n_r) = \delta(n_1)\cdots\delta(n_r). \end{aligned}$$

□

In order to apply Theorem 2.2, we introduce another definition. Let $\alpha \in \mathcal{M}_1$. A multiplication $P_\alpha(n_1, \ldots, n_r)$ is called a **principal function** equivalent to α if

$$P_\alpha(n_1, \ldots, n_r) = \begin{cases} \alpha(n_1) & \text{if } n_1 = \cdots = n_r, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1. Let $\Delta(n_1, n_2, \ldots, n_r) = \begin{cases} 1 & \text{if } n_2 \mid n_1, n_3 \mid n_1, \ldots, n_r \mid n_1, \\ 0 & \text{otherwise,} \end{cases} \in \mathcal{A}_r$. It is easily checked that $\Delta \in \mathcal{M}_r$ and $\Delta_1(n) = 1 = u(n)$, while each of its remaining component functions is the 1-dimensional identity function, i.e., $\Delta_i(n) = \delta(n)$ $(2 \le i \le r)$, so that Δ is a cardinal function. By Theorem 2.2, we have

$$\Delta(n_1, n_2, \ldots, n_r) = (u(n_1)\delta(n_2)\cdots\delta(n_r)) * K_\delta(n_1, n_2, \ldots, n_r),$$

where

$$\begin{aligned} &K_\delta(n_1, n_2, \ldots, n_r) = (\mu(n_1)\delta(n_2)\cdots\delta(n_r)) * \Delta\ (n_1, n_2, \ldots, n_r) \\ &= \sum_{d_1|n_1} \mu(n_1/d_1)\Delta(d_1, n_2, \ldots, n_r) = \begin{cases} 1 = u(n_1) & \text{if } n_1 = n_2 = \cdots = n_r, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

i.e., the cardinal component of Δ is a principal function equivalent to u, and so

$$\Delta(n_1, n_2, \ldots, n_r) = (u(n_1)\delta(n_2)\cdots\delta(n_r)) * P_u\,(n_1, n_2, \ldots, n_r).$$

Example 2. Our next example deals with two possible versions of a generalized Ramanujan sum of several variables.

A. For $r \ge 2$, the r-dimensional type I generalized Ramanujan sum (cf. [4] for the case of 2 variables) of order $\alpha \in \mathbb{C}$ is defined as

$$\begin{aligned} &C^{(\alpha)}(n_1, n_2, \ldots, n_r) := c^{(\alpha)}(n_1, n_2)\, c^{(\alpha)}(n_1, n_3)\cdots c^{(\alpha)}(n_1, n_r) \in \mathcal{M}_r \\ &= \sum_{d_2|\gcd(n_1,n_2)} d_2\mu_\alpha(n_2/d_2) \sum_{d_3|\gcd(n_1,n_3)} d_3\mu_\alpha(n_3/d_3)\cdots \sum_{d_r|\gcd(n_1,n_r)} d_r\mu_\alpha(n_r/d_r) \qquad (2.2) \\ &= \sum_{m_2 (\text{mod } n_2)} \mu_{\alpha-1}\left(\gcd(m_2, n_2)\right) e^{2\pi i m_2 n_1/n_2} \cdots \sum_{m_r (\text{mod } n_r)} \mu_{\alpha-1}\left(\gcd(m_r, n_r)\right) e^{2\pi i m_r n_1/n_r}. \end{aligned}$$

Its component functions are $C_1^{(\alpha)}(n) = 1 =: u(n),\ C_i^{(\alpha)}(n) = \mu_\alpha(n)\ \ (2 \le i \le r)$. Thus, its identical equation is

$$C^{(\alpha)}(n_1, n_2, \ldots, n_r) = (u(n_1)\mu_\alpha(n_2) \cdots \mu_\alpha(n_r)) * K_C\,(n_1, n_2, \ldots, n_r),$$

where

$$\begin{aligned} &K_C(n_1, n_2, \ldots, n_r) = \left(u^{-1}(n_1)\mu_\alpha^{-1}(n_2) \cdots \mu_\alpha^{-1}(n_r)\right) * C^{(\alpha)}(n_1, n_2, \ldots, n_r) \\ &= (\mu(n_1)\mu_{-\alpha}(n_2) \cdots \mu_{-\alpha}(n_r)) * C^{(\alpha)}(n_1, n_2, \ldots, n_r). \end{aligned}$$

To simplify K_C, we introduce the function $\eta(n_1, n_2) = \begin{cases} 1 & \text{if } n_2 \mid n_1, \\ 0 & \text{otherwise,} \end{cases} \in \mathcal{M}_2$. Thus, (2.2) becomes

$$C^{(\alpha)}(n_1, n_2, \ldots, n_r) = \sum_{d_2 | n_2} \eta(n_1, d_2)\mu_\alpha(n_2/d_2) \cdots \sum_{d_r | n_r} \eta(n_1, d_r)\mu_\alpha(n_r/d_r). \tag{2.3}$$

Using (2.3), we get

$$\begin{aligned} K_C(n_1, \ldots, n_r) &= \sum_{d_1 e_1 = n_1} \sum_{d_2 e_2 = n_2} \cdots \sum_{d_r e_r = n_r} \mu(d_1)\mu_{-\alpha}(d_2) \cdots \mu_{-\alpha}(d_r) C^{(\alpha)}(e_1, e_2, \ldots, e_r) \\ &= \sum_{d_1 e_1 = n_1} \mu(d_1) \sum_{d_2 i_2 j_2 = n_2} \mu_{-\alpha}(d_2)\eta(e_1, i_2)\mu_\alpha(j_2) \cdots \sum_{d_r e_r = n_r} \mu_{-\alpha}(d_r)\eta(e_1, i_r)\mu_\alpha(j_r) \\ &= \sum_{d_1 e_1 = n_1} \mu(d_1)\eta(e_1, n_2) \cdots \eta(e_1, n_r). \end{aligned}$$

B. The $r(\ge 2)$-dimensional type II generalized Ramanujan sum of order $\alpha \in \mathbb{C}$ is defined as

$$\begin{aligned} D^{(\alpha)}(n_1, n_2, \ldots, n_r) &:= \sum_{d | \gcd(n_1, n_2, \ldots, n_r)} d \cdot \mu_\alpha(n_2/d_2) \cdots \mu_\alpha(n_r/d_r) \in \mathcal{M}_r \\ &= \sum_{d_1 | n_1} \sum_{d_2 | n_2} \mu_\alpha(n_2/d_2) \cdots \sum_{d_r | n_r} \mu_\alpha(n_r/d_r) P_I(d_1, d_2, \ldots, d_r) \\ &= (u(n_1)\mu_\alpha(n_2) \cdots \mu_\alpha(n_r)) * P_I(n_1, n_2, \ldots, n_r), \end{aligned} \tag{2.4}$$

where $P_I(n_1, n_2, \ldots, n_r)$ is the principal function equivalent to the function $I(n) = n$, i.e.,

$$P_I(n_1, \ldots, n_r) = \begin{cases} I(n_1) = n_1 & \text{if } n_1 = \cdots = n_r, \\ 0 & \text{otherwise.} \end{cases}$$

The component functions of $D^{(\alpha)}$ are $D_1^{(\alpha)}(n) = 1 = u(n),\ D_i^{(\alpha)}(n) = \mu_\alpha(n)\ \ (2 \le i \le r)$. Thus, its identical equation is

$$D^{(\alpha)}(n_1, n_2, \ldots, n_r) = (u(n_1)\mu_\alpha(n_2) \cdots \mu_\alpha(n_r)) * K_D\,(n_1, n_2, \ldots, n_r), \tag{2.5}$$

where $K_D(n_1, n_2, \ldots, n_r) = (u^{-1}(n_1)\mu_\alpha^{-1}(n_2) \cdots \mu_\alpha^{-1}(n_r)) * D^{(\alpha)}(n_1, n_2, \ldots, n_r)$. Comparing (2.4) with (2.5), we get $K_D(n_1, n_2, \ldots, n_r) = P_I(n_1, n_2, \ldots, n_r)$.

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