Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 24, 2018, No. 4, 3–10
DOI: 10.7546/nntdm.2018.24.4.3-10

On a new arithmetic function

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To our friend and colleague Tony Shannon
for his $\varphi(164)$-th Anniversary!

Received: 30 September 2018 **Accepted:** 5 December 2018

Abstract: A new arithmetic function is introduced and its basic properties are studied. Some inequalities between the new and some other arithmetic functions are formulated and proved.
Keywords: Arithmetic function, Inequality.
2010 Mathematics Subject Classification: 11A25.

1 Introduction

In mathematical literature, there are definitions for a number of arithmetic functions. In the present paper, we use some of these arithmetic function, which for the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}, \tag{1}$$

$k, \alpha_1, \ldots, \alpha_k, k \geq 1$ being natural numbers and $p_1, \ldots, p_k$ being different primes, are defined by:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1),\ \varphi(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 1),\ \varphi(1) = 1,$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1},\ \sigma(1) = 1$$

$$\omega(n) = k, \omega(1) = 1$$

(see, e.g. [5, 7]). Also, we use the following notations for the above n:

$$\underline{mult}(1) = 1,\ \underline{mult}(n) = \prod_{i=1}^{k} p_i,$$

$$\underline{set}(1) = \emptyset,\ \underline{set}(n) = \{p_1, \ldots, p_k\}.$$

In the present paper, a new arithmetic function will be defined and some of its basic properties are studied.

2 Definition and properties of function SF

Let us define for the natural number n given by (1)

$$SF(1) = 1,$$

$$SF(n) = \frac{n}{\underline{mult}(n)} . \underline{mult}\left(\frac{n}{\underline{mult}(n)}\right). \tag{2}$$

We call this function "Squarefull factor".

Let for n there exist a set $I_n = \{i_1, \ldots, i_s\}$ for $0 \leq s \leq k$ with elements $i_1, \ldots, i_s$ satisfying the inequality $i_1 < \cdots < i_s$ and let $\alpha_{i_1} = \cdots = \alpha_{i_s} = 1$.

Obviously, if $I = \{1, 2, \ldots, k\}$, then $\alpha_{i_1}, \ldots, \alpha_{i_s} \geq 2$. Therefore, the following new representation of function S is possible.

Theorem 1. For the natural number n given by (1)

$$SF(n) = \prod_{\substack{j = 1 \\ j \notin I}}^{k} p_i^{\alpha_i}. \tag{3}$$

Proof. If $I = \emptyset$, then no power α will be equal to 1, i.e., for each i $(1 \leq i \leq k) : \alpha_i > 1$ or $\alpha_i \geq 2$. Therefore, p_i is a divisor of $\frac{n}{\underline{mult}(n)}$ with power $\alpha_i - 1$ and it has power 1 in $\underline{mult}\left(\frac{n}{\underline{mult}(n)}\right)$. Hence, in the right-hand side of (2) p_i has a degree α_i, as well as in the right-hand side of (3).

If $I \neq \emptyset$, i.e., if there exist $i_1, \ldots, i_s$ with the above property, the prime numbers $p_{i_1}, \ldots, p_{i_s}$ are not present in $\frac{n}{\underline{mult}(n)}$ and, therefore, they are not present in the right-hand side of (2), as well as in the right-hand side of (3).

Corollary 1. For each squarefree number n:

$$SF(n) = 1.$$

Corollary 2. For each natural number n:

$$SF(n) \leq n$$

and an equality exists only if n is squarefull.

Corollary 3. For every two natural numbers $m, k \geq 2$:

$$SF(m^k) = m^k.$$

Theorem 2. Function SF is multiplicative.

Proof. Let $(m, n) = 1$ be valid for the two natural numbers m, n. Then

$$\underline{mult}(mn) = \underline{mult}(m)\underline{mult}(n)$$

and

$$\begin{aligned} SF(mn) &= \frac{mn}{\underline{mult}(mn)}.\underline{mult}\left(\frac{mn}{\underline{mult}(mn)}\right) \\ &= \frac{m}{\underline{mult}(m)}.\frac{n}{\underline{mult}(n)}.\underline{mult}\left(\frac{m}{\underline{mult}(m)}.\frac{n}{\underline{mult}(n)}\right) \\ &= \frac{m}{\underline{mult}(m)}.\underline{mult}\left(\frac{m}{\underline{mult}(m)}\right).\frac{n}{\underline{mult}(n)}.\underline{mult}\left(\frac{n}{\underline{mult}(n)}\right) \\ &= SF(m)SF(n). \end{aligned}$$

A shorter proof of this theorem is based on the fact that functions $\underline{mult}(n)$ and

$$A(n) = \frac{n}{\underline{mult}(n)}$$

are multiplicative functions, and for $(a, b) = 1$ one has $(A(a), A(b)) = 1$. □

In the Table 1, the first 60 values of function SF are given.

n	$SF(n)$	n	$SF(n)$	n	$SF(n)$	n	$SF(n)$
1	1	16	16	31	1	46	1
2	1	17	1	32	32	47	1
3	1	18	9	33	1	48	16
4	4	19	1	34	1	49	49
5	1	20	4	35	1	50	25
6	1	21	1	36	36	51	1
7	1	22	1	37	1	52	4
8	8	23	1	38	1	53	1
9	9	24	8	39	1	54	27
10	1	25	25	40	8	55	1
11	1	26	1	41	1	56	8
12	4	27	27	42	1	57	1
13	1	28	4	43	1	58	1
14	1	29	1	44	4	59	1
15	1	30	1	45	9	60	4

Table 1. The first 60 values of function SF

3 Inequalities with participation of function SF

Theorem 3. For each natural number $n \geq 2$:

$$SF(n) \leq \psi(n) - \frac{n}{\underline{mult}(n)}. \tag{4}$$

Proof. Let n be a prime number. Then

$$SF(n) = 1 \leq n = (n+1) - 1 = \psi(n) - \frac{n}{\underline{mult}(n)}.$$

Let $n = p^m$, where p is a prime number and $m \geq 2$ is a natural number. Then

$$\begin{aligned} SF(n) &= \frac{p^m}{\underline{mult}(p^m)}.\underline{mult}\left(\frac{p^m}{\underline{mult}(p^m)}\right) \\ &= \frac{p^m}{p}.\underline{mult}\left(\frac{p^m}{p}\right) \\ &= p^{m-1}.\underline{mult}(p^{m-1}) = p^m = p^{m-1}(p+1) - p^{m-1} \\ &= \psi(n) - \frac{n}{\underline{mult}(n)}. \end{aligned}$$

Let us assume that (4) be valid for some natural number n and let p be a prime number. If $p \notin \underline{set}(n)$, then from Theorem 2 it follows that

$$\begin{aligned} SF(np) &= SF(n)SF(p) = SF(n) \\ &\leq \psi(n) - \frac{n}{\underline{mult}(n)} \\ &\leq p\left(\psi(n) - \frac{n}{\underline{mult}(n)}\right) \\ &\leq \psi(n)(p+1) - \frac{np}{\underline{mult}(n)} \\ &\leq \psi(np) - \frac{np}{\underline{mult}(n)p} \\ &\leq \psi(np) - \frac{np}{\underline{mult}(np)}. \end{aligned}$$

If $p \in \underline{set}(n)$, then $\underline{mult}(np) = \underline{mult}(n)$, and

$$\begin{aligned} SF(np) &= \frac{np}{\underline{mult}(np)}.\underline{mult}\left(\frac{np}{\underline{mult}(np)}\right) \\ &= \frac{np}{\underline{mult}(n)}.\underline{mult}\left(\frac{np}{\underline{mult}(n)}\right) \\ &= p.\frac{n}{\underline{mult}(n)}.\underline{mult}\left(\frac{n}{\underline{mult}(n)}\right) \\ &= SF(n)p \leq p\left(\psi(n) - \frac{n}{\underline{mult}(n)}\right) \end{aligned}$$

$$= \psi(np) - \frac{np}{\underline{mult}(n)} = \psi(np) - \frac{np}{\underline{mult}(np)}.$$

Another proof of this theorem is the following. Using the value of $\psi(n)$ given by the prime factorization of n per (1), the inequality (4) can be reduced to:

$$\underline{mult}(p_1^{\alpha_1 - 1} \cdots p_r^{\alpha_r - 1)} + 1 \leq (p_1 + 1) \cdots (p_r + 1). \tag{5}$$

As the left-hand side of (5) is less than or equal to $p_1 \cdots p_r$, inequality (5) follows by the following inequality:

$$p_1 \cdots p_r + 1 \leq (p_1 + 1) \cdots (p_r + 1). \tag{6}$$

Inequality (6) is well-known, and follows easily, e.g., by mathematical induction. There is an equality in (6) only for $r = 1$.

It is immediate that, there is an equality in relation (4) only if $n = p^a$, where p is a prime and $a \geq 2$ (i.e., a squarefull number, with a single prime divisor). $\square$

Corollary 4. For each natural number n:

$$SF(n) \leq \sigma(n) - \frac{n}{\underline{mult}(n)}.$$

From Corollary 4 one gets also

$$SF(n) \leq \frac{n^2}{\varphi(n)} - \frac{n}{\underline{mult}(n)}.$$

Let $\min(n) = \min(\underline{set}(n))$. Then the following theorem is valid.

Theorem 4. For each natural number $n \geq 2$:

$$SF(n) < \varphi(n) \left(1 - \frac{1}{\min(n)}\right)^{-\omega(n)}. \tag{7}$$

Proof. We see directly that

$$\frac{SF(n)}{\varphi(n)} = \frac{\frac{n}{\underline{mult}(n)} . \underline{mult}\left(\frac{n}{\underline{mult}(n)}\right)}{\prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)} = \frac{\underline{mult}\left(\frac{n}{\underline{mult}(n)}\right)}{\prod_{i=1}^{k} (p_i - 1)}$$

$$\leq \frac{\prod_{i=1}^{k} p_i}{\prod_{i=1}^{k} (p_i - 1)} = \prod_{i=1}^{k} \frac{1}{1 - \frac{1}{p_i}} \leq \left(1 - \frac{1}{\min(n)}\right)^{-\omega(n)}. \qquad \square$$

Theorem 5. For each natural number n:

$$SF(n) < \frac{\pi^2}{6} . \frac{\varphi(n)\psi(n)}{n} \tag{8}$$

Proof. From definition of function ψ we have

$$\frac{n}{\psi(n)} = \prod_{i=1}^{k} \frac{1}{1+\frac{1}{p_i}}.$$

Now, we obtain

$$\prod_{i=1}^{k} \frac{1}{1-\frac{1}{p_i}} = \prod_{i=1}^{k} \frac{\frac{1}{1+\frac{1}{p_i}}}{\frac{1}{1-\frac{1}{p_i^2}}}.$$

Let $\mathcal{P}$ be the set of all primes. Then

$$\prod_{i=1}^{k} \frac{1}{1-\frac{1}{p_i^2}} < \prod_{p\in\mathcal{P}} \frac{1}{1-\frac{1}{p^2}} = \zeta(2) = \frac{\pi^2}{6}.$$

Therefore, inequality (8) follows.

This inequality is the best possible, as can be seen by taking $n = (p_1 \cdots p_r)^2$, where now $p_i \in \mathcal{P}$ is the i-th prime number, then for this particular n one has

$$\frac{nSF(n)}{\varphi(n)\psi(n)} = \prod_{i=1}^{r} \frac{1}{1-\frac{1}{p_i^2}}$$

having the limit, as r tends to ∞, exactly $\zeta(2) = \frac{\pi^2}{6}$.

In [1, 2, 4] the following function is defined for n from (1):

$$\delta(n) = \sum_{i=1}^{k} \alpha_i p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k}.$$

Theorem 6. For each natural number n:

$$SF(n) \le \delta(n). \tag{10}$$

Proof. Let n be a prime number. Then

$$SF(n) = 1 = \delta(n).$$

Let us assume that (10) be valid for some natural number n and let p be a prime number.

If $p \notin \underline{set}(n)$, then from Theorem 2 it follows that

$$SF(np) = SF(n)SF(p) = SF(n) \le \delta(n) \le \delta(n)p + n = \delta(np).$$

If $p \in \underline{set}(n)$, then as in the proof of Theorem 3:

$$SF(np) = SF(n)p \le p\delta(n) < \delta(n)p + n = \delta(np). \qquad \square$$

In [3] the following function is defined for n from (1):

$$RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1} = \frac{n}{\underline{mult}(n)}.$$

Some of its properties are discussed in [6]. From (2) it follows directly that for each natural number n:

$$RF(n) \le SF(n).$$

4 Conclusion

In near future, other propertiers of function SF will be discussed. For example, at the moment an **Open problem** is: What is the relation between function SF and the other arithmetic functions. On the other hand, we can introduce the following new arithmetic function.

$$t(n) = \underline{mult}(\varphi(n)). \tag{11}$$

One has $t(1) = 1, t(2) = 1, t(2^k) = 2$ for any $k \geq 2$ and

$$t(p^k) = p.\underline{mult}(p-1)$$

for any odd prime p. Particularly, one has

$$t(p^k) \geq 2p.$$

On the other hand, one has $t(n) \leq \varphi(n) \leq n-1$ for any $n \geq 2$.

Now, it is immediate that,

$$\underline{mult}(ab) \leq \underline{mult}(a).\underline{mult}(b)$$

for any $a, b \geq 1$.

Let $(u, v) = 1$. Then we get

$$t(uv) = \underline{mult}(\varphi(u).\varphi(v)) \leq \underline{mult}(\varphi(u)).\underline{mult}(\varphi(v)),$$

by the above property. Therefore, we get:

$$t(uv) \leq t(u).t(v)$$

for any $(u, v) = 1$.

We have used also that the function φ is multiplicative. Using relation (11), we get that for the prime factorization (1) from the article, one has:

$$t(n) \leq \prod_{p/n} p.\underline{mult}(p-1).$$

Perhaps, other properties of this function could be established, too.

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