On a new arithmetic function

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To our friend and colleague Tony Shannon
for his $\varphi(164)$-th Anniversary!

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Abstract: A new arithmetic function is introduced and its basic properties are studied. Some inequalities between the new and some other arithmetic functions are formulated and proved.

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1 Introduction

In mathematical literature, there are definitions for a number of arithmetic functions. In the present paper, we use some of these arithmetic function, which for the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

$k, \alpha_1, \ldots, \alpha_k, k \geq 1$ being natural numbers and $p_1, \ldots, p_k$ being different primes, are defined by:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i - 1), \varphi(1) = 1,$$
\[
\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 1), \quad \varphi(1) = 1,
\]
\[
\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}, \quad \sigma(1) = 1
\]
\[
\omega(n) = k, \quad \omega(1) = 1
\]

(see, e.g. [5, 7]). Also, we use the following notations for the above \( n \):

\[
\text{mult}(1) = 1, \quad \text{mult}(n) = \prod_{i=1}^{k} p_i,
\]
\[
\text{set}(1) = \emptyset, \quad \text{set}(n) = \{p_1, \ldots, p_k\}.
\]

In the present paper, a new arithmetic function will be defined and some of its basic properties are studied.

2 Definition and properties of function \( SF \)

Let us define for the natural number \( n \) given by (1)

\[
SF(1) = 1,
\]
\[
SF(n) = \frac{n}{\text{mult}(n)} \text{mult} \left( \frac{n}{\text{mult}(n)} \right). 
\]  \hspace{1cm} (2)

We call this function “Squarefull factor”.

Let for \( n \) there exist a set \( I_n = \{i_1, \ldots, i_s\} \) for \( 0 \leq s \leq k \) with elements \( i_1, \ldots, i_s \) satisfying the inequality \( i_1 < \cdots < i_s \) and let \( \alpha_{i_1} = \cdots = \alpha_{i_s} = 1 \).

Obviously, if \( I = \{1, 2, \ldots, k\} \), then \( \alpha_{i_1}, \ldots, \alpha_{i_s} \geq 2 \). Therefore, the following new representation of function \( S \) is possible.

**Theorem 1.** For the natural number \( n \) given by (1)

\[
SF(n) = \prod_{j=1, j \neq i}^{k} p_i^{\alpha_i}. 
\]  \hspace{1cm} (3)

**Proof.** If \( I = \emptyset \), then no power \( \alpha \) will be equal to 1, i.e., for each \( i \) \((1 \leq i \leq k) : \alpha_i > 1 \) or \( \alpha_i \geq 2 \). Therefore, \( p_i \) is a divisor of \( \frac{n}{\text{mult}(n)} \) with power \( \alpha_i - 1 \) and it has power 1 in \( \text{mult} \left( \frac{n}{\text{mult}(n)} \right) \). Hence, in the right-hand side of (2) \( p_i \) has a degree \( \alpha_i \), as well as in the right-hand side of (3).

If \( I \neq \emptyset \), i.e., if there exist \( i_1, \ldots, i_s \) with the above property, the prime numbers \( p_{i_1}, \ldots, p_{i_s} \) are not present in \( \frac{n}{\text{mult}(n)} \) and, therefore, they are not present in the right-hand side of (2), as well as in the right-hand side of (3).

**Corollary 1.** For each squarefree number \( n \): \[
SF(n) = 1.
\]
Corollary 2. For each natural number \( n \):
\[
SF(n) \leq n
\]
and an equality exists only if \( n \) is squarefull.

Corollary 3. For every two natural numbers \( m, k \geq 2 \):
\[
SF(mk) = mk.
\]

Theorem 2. Function \( SF \) is multiplicative.

Proof. Let \((m, n) = 1 \) be valid for the two natural numbers \( m, n \). Then
\[
mult(mn) = mult(m)mult(n)
\]
and
\[
SF(mn) = \frac{mn}{mult(mn)} \cdot mult\left(\frac{mn}{mult(mn)}\right)
\]
\[
= \frac{m}{mult(m)} \cdot \frac{n}{mult(n)} \cdot mult\left(\frac{m}{mult(m)}\right) \cdot mult\left(\frac{n}{mult(n)}\right)
\]
\[
= SF(m)SF(n).
\]

A shorter proof of this theorem is based on the fact that functions \( mult(n) \) and
\[
A(n) = \frac{n}{mult(n)}
\]
are multiplicative functions, and for \((a, b) = 1 \) one has \((A(a), A(b)) = 1\). \(\square\)

In the Table 1, the first 60 values of function \( SF \) are given.

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Table 1. The first 60 values of function \( SF \)
3 Inequalities with participation of function \( SF \)

**Theorem 3.** For each natural number \( n \geq 2 \):

\[
SF(n) \leq \psi(n) - \frac{n}{\text{mult}(n)}.
\]

(4)

**Proof.** Let \( n \) be a prime number. Then

\[
SF(n) = 1 \leq n = (n + 1) - 1 = \psi(n) - \frac{n}{\text{mult}(n)}.
\]

Let \( n = p^m \), where \( p \) is a prime number and \( m \geq 2 \) is a natural number. Then

\[
SF(n) = \frac{p^m}{\text{mult}(p^m)} \cdot \text{mult}\left(\frac{p^m}{\text{mult}(p^m)}\right)
= \frac{p^m}{p} \cdot \text{mult}\left(\frac{p^m}{p}\right)
= p^{m-1} \cdot \text{mult}(p^{m-1}) = p^m = p^m(p + 1) - p^{m-1}
= \psi(n) - \frac{n}{\text{mult}(n)}.
\]

Let us assume that (4) be valid for some natural number \( n \) and let \( p \) be a prime number.

If \( p \not\in \text{set}(n) \), then from Theorem 2 it follows that

\[
SF(np) = SF(n)SF(p) = SF(n)
\leq \psi(n) - \frac{n}{\text{mult}(n)}
\leq p \left( \psi(n) - \frac{n}{\text{mult}(n)} \right)
\leq \psi(n)(p + 1) - \frac{np}{\text{mult}(n)}
\leq \psi(np) - \frac{np}{\text{mult}(np)}
\leq \psi(np) - \frac{np}{\text{mult}(n)p}.
\]

If \( p \in \text{set}(n) \), then \( \text{mult}(np) = \text{mult}(n) \), and

\[
SF(np) = \frac{np}{\text{mult}(np)} \cdot \text{mult}\left(\frac{np}{\text{mult}(np)}\right)
= \frac{np}{\text{mult}(n)} \cdot \text{mult}\left(\frac{np}{\text{mult}(n)}\right)
= p \cdot \frac{n}{\text{mult}(n)} \cdot \text{mult}\left(\frac{np}{\text{mult}(n)}\right)
= SF(n)p \leq p \left( \psi(n) - \frac{n}{\text{mult}(n)} \right)
\]
\[ = \psi(np) - \frac{np}{\text{mult}(n)} = \psi(np) - \frac{np}{\text{mult}(np)}. \]

Another proof of this theorem is the following. Using the value of \( \psi(n) \) given by the prime factorization of \( n \) per (1), the inequality (4) can be reduced to:

\[ \text{mult}(p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1}) + 1 \leq (p_1 + 1) \cdots (p_r + 1). \quad (5) \]

As the left-hand side of (5) is less than or equal to \( p_1 \cdots p_r \), inequality (5) follows by the following inequality:

\[ p_1 \cdots p_r + 1 \leq (p_1 + 1) \cdots (p_r + 1). \quad (6) \]

Inequality (6) is well-known, and follows easily, e.g., by mathematical induction. There is an equality in (6) only for \( r = 1 \).

It is immediate that, there is an equality in relation (4) only if \( n = p^a \), where \( p \) is a prime and \( a \geq 2 \) (i.e., a squarefull number, with a single prime divisor). \( \square \)

**Corollary 4.** For each natural number \( n \):

\[ SF(n) \leq \sigma(n) - \frac{n}{\text{mult}(n)}. \]

From Corollary 4 one gets also

\[ SF(n) \leq \frac{n^2}{\varphi(n)} - \frac{n}{\text{mult}(n)}. \]

Let \( \min(n) = \min(\text{set}(n)) \). Then the following theorem is valid.

**Theorem 4.** For each natural number \( n \geq 2 \):

\[ SF(n) < \varphi(n) \left( 1 - \frac{1}{\min(n)} \right)^{-\omega(n)}. \quad (7) \]

**Proof.** We see directly that

\[
\begin{align*}
SF(n) &= \frac{\text{mult}(n)}{\varphi(n)} \cdot \text{mult} \left( \frac{n}{\text{mult}(n)} \right) = \frac{\text{mult} \left( \frac{n}{\text{mult}(n)} \right)}{\prod_{i=1}^{k} (p_i - 1)} \\
&\leq \frac{\prod_{i=1}^{k} p_i}{\prod_{i=1}^{k} (p_i - 1)} = \prod_{i=1}^{k} \frac{1}{1 - \frac{1}{p_i}} \leq \left( 1 - \frac{1}{\min(n)} \right)^{-\omega(n)}. \quad \square
\end{align*}
\]

**Theorem 5.** For each natural number \( n \):

\[ SF(n) < \frac{\pi^2 \varphi(n) \psi(n)}{6n}. \quad (8) \]
Proof. From definition of function $\psi$ we have

\[ \frac{n}{\psi(n)} = \prod_{i=1}^{k} \frac{1}{1 + \frac{1}{p_i}}. \]

Now, we obtain

\[ \prod_{i=1}^{k} \frac{1}{1 - \frac{1}{p_i}} = \prod_{i=1}^{k} \frac{1 + \frac{1}{p_i}}{1 - \frac{1}{p_i}}. \]

Let $\mathcal{P}$ be the set of all primes. Then

\[ \prod_{i=1}^{k} \frac{1}{1 - \frac{1}{p_i}} < \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{1}{p^2}} = \zeta(2) = \frac{\pi^2}{6}. \]

Therefore, inequality (8) follows.

This inequality is the best possible, as can be seen by taking $n = (p_1 \cdots p_r)^2$, where now $p_i \in \mathcal{P}$ is the $i$-th prime number, then for this particular $n$ one has

\[ \frac{nSF(n)}{\varphi(n)\psi(n)} = \prod_{i=1}^{r} \frac{1}{1 - \frac{1}{p_i}} \]

having the limit, as $r$ tends to $\infty$, exactly $\zeta(2) = \frac{\pi^2}{6}$.

In [1, 2, 4] the following function is defined for $n$ from (1):

\[ \delta(n) = \sum_{i=1}^{k} \alpha_i p_i^{\alpha_i} \cdot \cdots \cdot p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k}. \]

**Theorem 6.** For each natural number $n$:

\[ SF(n) \leq \delta(n). \] (10)

**Proof.** Let $n$ be a prime number. Then

\[ SF(n) = 1 = \delta(n). \]

Let us assume that (10) be valid for some natural number $n$ and let $p$ be a prime number.

If $p \notin \text{set}(n)$, then from Theorem 2 it follows that

\[ SF(np) = SF(n)SF(p) = SF(n) \leq \delta(n) \leq \delta(n)p + n = \delta(np). \]

If $p \in \text{set}(n)$, then as in the proof of Theorem 3:

\[ SF(np) = SF(n)p \leq p\delta(n) < \delta(n)p + n = \delta(np). \]

In [3] the following function is defined for $n$ from (1):

\[ RF(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i - 1}}{\text{mult}(n)} = \frac{n}{\text{mult}(n)}. \]

Some of its properties are discussed in [6]. From (2) it follows directly that for each natural number $n$:

\[ RF(n) \leq SF(n). \]
4 Conclusion

In near future, other properties of function $SF$ will be discussed. For example, at the moment an Open problem is: What is the relation between function $SF$ and the other arithmetic functions. On the other hand, we can introduce the following new arithmetic function.

$$t(n) = \text{mult}(\varphi(n)).$$

One has $t(1) = 1, t(2) = 1, t(2^k) = 2$ for any $k \geq 2$ and

$$t(p^k) = p \cdot \text{mult}(p - 1)$$

for any odd prime $p$. Particularly, one has

$$t(p^k) \geq 2p.$$

On the other hand, one has $t(n) \leq \varphi(n) \leq n - 1$ for any $n \geq 2$. Now, it is immediate that,

$$\text{mult}(ab) \leq \text{mult}(a) \cdot \text{mult}(b)$$

for any $a, b \geq 1$.

Let $(u, v) = 1$. Then we get

$$t(uv) = \text{mult}(\varphi(u) \cdot \varphi(v)) \leq \text{mult}(\varphi(u)) \cdot \text{mult}(\varphi(v)),$$

by the above property. Therefore, we get:

$$t(uv) \leq t(u) \cdot t(v)$$

for any $(u, v) = 1$.

We have used also that the function $\varphi$ is multiplicative. Using relation (11), we get that for the prime factorization (1) from the article, one has:

$$t(n) \leq \prod_{p|n} p \cdot \text{mult}(p - 1).$$

Perhaps, other properties of this function could be established, too.

References


