Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 24, 2018, No. 3, 95–102
DOI: 10.7546/nntdm.2018.24.3.95-102

On repdigits as product of consecutive Lucas numbers

Nurettin Irmak[1] and Alain Togbé[2]

[1] Mathematics Department, Art and Science Faculty
Niğde Ömer Halisdemir University, Niğde, Turkey
e-mails: irmaknurettin@gmail.com, nirmak@ohu.edu.tr

[2] Department of Mathematics, Purdue University Northwest
1401 S. U. S. 421., Westville, IN 46391, United States
e-mail: atogbe@pnw.edu

Received: 15 November 2017 **Accepted:** 22 July 2018

Abstract: Let $(L_n)_{n\geq 0}$ be the Lucas sequence. D. Marques and A. Togbé [7] showed that if $F_n \dots F_{n+k-1}$ is a repdigit with at least two digits, then $(k, n) = (1, 10)$, where $(F_n)_{\geq 0}$ is the Fibonacci sequence. In this paper, we solve the equation

$$L_n \dots L_{n+k-1} = a\left(\frac{10^m - 1}{9}\right),$$

where $1 \leq a \leq 9$, $n,\ k \geq 2$ and m are positive integers.
Keywords: Lucas numbers, Repdigits.
2010 Mathematics Subject Classification: 11A63, 11B39, 11B50.

1 Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by the relation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ and with $F_0 = 0$, $F_1 = 1$. Its companion sequence is known as Lucas sequence $(L_n)_{n\geq 0}$ that satisfies the same relation with Fibonacci sequence together with the initial conditions $L_0 = 2$ and $L_1 = 1$. These numbers have very amazing properties (for see details, we refer the book of Koshy [3]).

Finding special properties in these sequences is a very interesting problem. The most famous one is given by Bugeaud et al. [1] as that the only perfect powers in Fibonacci sequence are

$0, 1, 8$ and 144. Luca and Shorey [6] proved that product of consecutive Fibonacci numbers is not a perfect power of exponent larger than one of an integer except the trivial case $F_1F_2 = 1$.

If a positive integer has only one distinct digit in its decimal expansion, then we call it "repdigit". Obviously, such a number has the form $a\,(10^m - 1)\,/9$, for some $m \geq 1$ and $1 \leq a \leq 9$. It is natural to ask, which numbers are repdigits in Fibonacci and Lucas sequences? This question was answered by Luca [5] in 2000 by showing that 55 is the largest repdigit Fibonacci number and 11 is the largest repdigit Lucas number. Recently, Marques and Togbé [7] proved the following result.

Theorem 1. *The only solution of the Diophantine equation*

$$F_n \cdots F_{n+(k-1)} = a\left(\frac{10^m - 1}{9}\right), \tag{1}$$

in positive integers n, k, m, a, *with* $1 \leq a \leq 9$ *and* $m > 1$ *is* $(n, k, m, a) = (10, 1, 2, 5)$.

For the proof of the above theorem, they used mathematical induction, Fibonacci recurrence pattern, congruence properties, etc. But, the main point of their proof is the identity $5|F_{5n}\,(n \geq 0)$.

The aim of this paper is to study a similar problem but in the case of Lucas numbers. Our main result is as follows.

Theorem 2. *The quadruple* $(n, k, m, a) = (4, 2, 2, 7)$ *is the only solution of the Diophantine equation*

$$L_n \ldots L_{n+k-1} = a\left(\frac{10^m - 1}{9}\right), \tag{2}$$

for some $m \geq 1,\ k \geq 2$ *and* $1 \leq a \leq 9$ *being integers.*

In order to solve the equation, we use the definition of p-adic order of an integer, linear forms in logarithms à la Baker, and congruence properties.

2 Auxiliary results

The p-adic order of r is the exponent of the highest power of a prime p which divides r. We denote it by $\nu_p\,(r)$. Now, we recall a result of Lengyel [4] on the 2-adic order of a Lucas number.

Lemma 1. *For* $n \geq 0$ *integer, then*

$$\nu_2\,(L_n) = \begin{cases} 0, & n \equiv 1, 2 \pmod 3 \\ 2, & n \equiv 3 \pmod 6 \\ 1, & n \equiv 0 \pmod 6. \end{cases}$$

The Binet formula for a Lucas number is

$$L_n = \alpha^n + \beta^n,$$

where $\alpha\,(> 1)$ and $\beta\,(< 1)$ are the roots of the characteristic equation $x^2 - x - 1 = 0$. Moreover, we have

$$\alpha^{n-1} < L_n < \alpha^{n+1}, \quad \text{for } n \geq 0. \tag{3}$$

Since we use Baker method for our proof, we give two lemmata due to Matveev [8] (and Theorem 9.4 of [1]), Dujella and Pethö [2]. It is well-known that the logarithmic height of an algebraic number η is defined as

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{\left|\eta^{(i)}, 1\right|\right\}\right)\right),$$

where d is the degree of η over $\mathbb{Q}$ and $\left(\eta^{(i)}\right)_{1\le i\le d}$ are the conjugates of η over $\mathbb{Q}$, and a_0 is the leading coefficient of the irreducible polynomial of η.

Lemma 2. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, b_2, \ldots, b_t$ nonzero rational integers. Put*

$$B \ge \max\{|b_1|, |b_2|, \ldots, |b_t|\},$$

and

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.$$

Let $A_1, \ldots, A_t$ be positive real numbers such that

$$A_i \ge \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \qquad i = 1, \ldots, t.$$

Then, assuming that $\Lambda \ne 0$, we have

$$|\Lambda| > \exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2 \times (1 + \log D)(1 + \log B) A_1 \ldots A_t\right).$$

Lemma 3. *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$ and $\epsilon = \| \mu q \| - M \| \gamma q \|$, where μ is a real number and $\| \cdot \|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-m}$$

in positive integers m and n with

$$\frac{\log(Aq/\epsilon)}{\log B} \le m < M.$$

3 Proof of Theorem 2

3.1 The congruence method

If $m = 2$, then we get the solution

$$L_4 L_5 = 7 \cdot 11 = 7\left(\frac{10^2 - 1}{9}\right).$$

Assume that $k \ge 9$ and $m \ge 4$. Then, we have

$$\nu_2(L_n \ldots L_{n+k-1}) = \nu_2(L_n) + \cdots + \nu_2(L_{n+k-1}) \ge 4.$$

Since $\nu_2\left(a\left(\frac{10^m-1}{9}\right)\right) \le 3$, for $1 \le a \le 9$ an integer, then it yields an absurdity. So it follows that $k \le 8$.

If we focus on the 2-adic order of Lucas numbers and $a\left(\frac{10^m-1}{9}\right)$ in equation (2), then we have the several possible equations. We give the following tables containing the results obtained after taking the left and the right sides of equation (2) modulo t. As an example, it is obvious to see that $L_{6n}L_{6n+1} \equiv 2 \pmod 5$. Namely, $\{L_k\}_{k\geq 0}$ has period 4 modulo 5 with the period being $2, 1, 3, 4$. Since $6n$ is even, it follows that either $6n$ is a multiple of 4, so then the two consecutive Lucas numbers are at the beginning of the period, namely the residues $2,\ 1$ with $2 \cdot 1 \equiv 2 \pmod 5$, or $6n \equiv 2 \pmod 4$, in which case the Lucas numbers are the last two, namely the residues $3, 4$ with $3 \cdot 4 \equiv 2 \pmod 5$.

It is obvious that $6\frac{10^m-1}{9} \equiv 1 \pmod 5$. Therefore, we arrive at a contradiction. Since the other cases can be proved by induction, we omit them. The sign † means a contradiction.

- Suppose that $n \equiv 0 \pmod 6$.

k	a	t	$L_n \dots L_{n+k-1} \pmod t$	$a\left(\frac{10^m-1}{9}\right) \pmod t$	Result
2	2	16	2 or 10	14	†
2	6	5	2	1	†
4	8	10	4	8	†
3	2	5	1 or 4	2	†
3	6	8	6	2	†
5	8	25	7 or 18	13	†
6	8	25	$18, 3$ or 23	13	†

- Suppose that $n \equiv 1 \pmod 6$. Then, we have:

k	a	t	$L_n \dots L_{n+k-1} \pmod t$	$a\left(\frac{10^m-1}{9}\right) \pmod t$	Result
2	1	8	3	7	†
2	3	8	3	5	†
2	5	10	3	5	†
2	7	10	3	7	†
2	9	10	3	9	†
3	4	10	2 or 8	4	†
4	4	39	$24, 33, 33, 24, 6, 27, 6$	$0, 4, 5, 15, 37, 23$	†
5	4	37	$19, 34, 28, 12, 11, 24, 8, 17, 3,$ $17, 8, 24, 11, 12, 28, 34, 19, 36,$ $1, 18, 3, 9, 25, 26, 13, 29, 20, 34,$ $20, 29, 13, 26, 25, 9, 3, 18, 1, 36$	$0, 4, 7$	†
6	8	30	12	$18, 8, 28$	†

• If $n \equiv 2 \pmod 6$, then

k	a	t	$L_n \dots L_{n+k-1} \pmod t$	$a\left(\frac{10^m-1}{9}\right) \pmod t$	Result
2	4	10	2	4	†
3	4	39	$21, 33, 36, 33, 21, 6$ $33, 18, 6, 3, 6, 18, 33, 6$	$15, 37, 23, 0, 4, 5$	†
4	4	39	$6, 24, 33, 33, 24, 6, 27$	$0, 4, 5, 15, 37, 23$	†
5	8	39	$21, 27, 36, 21, 36, 27, 21$ $18, 12, 3, 18, 3, 12, 18$	$0, 8, 10, 30, 35, 7$	†
6	8	39	$21, 15, 12, 6, 6, 12, 15$	$0, 8, 10, 30, 35, 7$	†

• Now, assume that $n \equiv 3 \pmod 6$. Then, we get following table.

k	a	t	$L_n \dots L_{n+k-1} \pmod t$	$a\left(\frac{10^m-1}{9}\right) \pmod t$	Result
2	4	5	3	4	†
3	4	5	2	4	†
4	8	10	4	8	†
5	8	10	4 or 6	8	†
6	8	10	2	8	†

• Assume that $n \equiv 4 \pmod 6$. Then, we obtain:

k	a	t	$L_n \dots L_{n+k-1} \pmod t$	$a\left(\frac{10^m-1}{9}\right) \pmod t$	Result
2	1	5	2	1	†
2	3	20	17	13	†
2	5	5	2	0	†
2	7	8	5	1	†
2	9	9	5 or 6	0	†
3	2	8	2	6	†
3	6	–	–	–	possible
4	2	20	14	2	†
4	6	20	14	6	†
5	2	25	$18, 7$	22	†
5	6	25	$18, 7$	16	†
6	8	25	$18, 23, 3$	13	†

• Finally, if $n \equiv 5 \pmod 6$, then

k	a	t	$L_n \dots L_{n+k-1} \pmod t$	$a\left(\frac{10^m-1}{9}\right) \pmod t$	Result
2	2	20	18	2	†
2	6	24	$6, 22$	18	†
3	2	25	$13, 12, 8, 2, 8,$ $12, 13, 17, 23, 17$	22	†
3	6	16	14	10	†
4	2	15	9	$12, 2$ or 7	†
4	6	15	9	6	†
5	8	25	$1, 24$	13	†
6	8	25	$7, 2, 22$	13	†

From all the tables, we see that the equation $L_nL_{n+1}L_{n+2} = \frac{6}{9}(10^m - 1)$ is possible if $n \equiv 4 \pmod 6$. We solve this equation in the next subsection by Baker's method.

3.2 The equation $L_nL_{n+1}L_{n+2} = \frac{6}{9}(10^m - 1)$

In this subsection, we prove that the equation

$$L_nL_{n+1}L_{n+2} = \frac{6}{9}(10^m - 1) = \frac{2}{3}(10^m - 1) \tag{4}$$

has no solution with positive integers n and $m \geq 3$. Combining the Binet formula for Lucas numbers with the fact $L_nL_{n+1}L_{n+2} = L_{3n+3} + (-1)^n 2L_{n+1}$, we get

$$\alpha^{3n+3} - \frac{2}{3}10^m = \frac{-2}{3} - \beta^{3n+3} - (-1)^n\left(2\alpha^{n+1} + 2\beta^{n+1}\right).$$

Thus, we obtain

$$\begin{aligned}\left|1 - \frac{2}{3}10^m\alpha^{-(3n+3)}\right| &\leq \frac{2}{3\alpha^{3n+3}} + \frac{|\beta|^{3n+3}}{\alpha^{3n+3}} + \frac{2}{\alpha^{2n+2}} + \frac{2|\beta|^{n+1}}{\alpha^{3n+3}} \\ &< \frac{4}{\alpha^{3n+3}} + \frac{2}{\alpha^{2n+2}} < \frac{6}{\alpha^{2n+2}}.\end{aligned} \tag{5}$$

Let

$$\Lambda := \frac{2}{3}\alpha^{-(3n+3)}10^m - 1.$$

In order to apply Lemma 2, we take

$$\gamma_1 := \frac{2}{3},\ \gamma_2 := \alpha,\ \gamma_3 := 10,\ b_1 := 1,\ b_2 := 3n+3,\ b_3 := m.$$

For this choice, we have $D = 2,\ t = 3,\ B = 3n + 3,\ A_1 := 0.16,\ A_2 := 0.5$, and $A_3 := 2.31$. As $\alpha,\ 10$, and $2/3$ are multiplicatively independent, we have $\Lambda \neq 0$. We combine Lemma 2 and the inequality (5) to obtain

$$\exp\left(K\left(1 + \log\left(3n+3\right)\right)\right) < |\Lambda| < \frac{6}{\alpha^{2n+2}}, \tag{6}$$

where $K := -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4\,(1+\log 2) \cdot 0.16 \cdot 0.5 \cdot 2.31$. Inequality (6) yields that

$$n < 5.87 \cdot 10^{12}.$$

By the estimates for Lucas numbers given by (3) and equation (4), we have $10^m < \alpha^{3n+6}$ and then

$$\left|1 - \frac{2}{3}10^m \alpha^{-(3n+3)}\right| < \frac{6}{\alpha^{2n+2}} < \frac{73}{(10^{2/3})^m}.$$

Let $z := m\log 10 - (3n+3)\log\alpha + \log\frac{2}{3}$. Thus,

$$|1-e^z| < \frac{73}{(10^{2/3})^m}$$

holds. It is obvious that $z \neq 0$. If $z > 0$, then

$$0 < z \leq |1-e^z| < \frac{73}{(10^{2/3})^m}.$$

Otherwise ($z < 0$), we get

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}\,|1-e^z| < \frac{146}{(10^{2/3})^m},$$

where we use the fact $|1-e^z| < \frac{1}{2}$. In any case, we obtain that

$$\begin{aligned} 0 &< \left|m\log 10 - (3n+3)\log\alpha + \log\frac{2}{3}\right| \\ &< \frac{146}{(10^{2/3})^m} < \frac{146}{(4.6)^m}. \end{aligned}$$

Dividing by $3\log\alpha$, we get

$$|m\gamma - n + \mu| < 102 \cdot (4.6)^{-m}.$$

with $\gamma := \frac{\log 10}{3\log\alpha}$ and $\mu := \frac{\log\left(\frac{2}{3}\right)}{3\log\alpha}$. Let q_t be the denominator of the t-th convergent of the continued fraction of γ. Taking $M := 5.87 \cdot 10^{12}$, we have

$$q_{32} = 109143857145934 > 6M,$$

and then $\epsilon := \|\mu q_{32}\| - M\,\|\gamma q_{32}\| > 0$. The conditions of Lemma 3 are fulfilled for $A := 102$ and $B := 4.6$. Then, there are no solutions for the interval

$$\left[\left\lfloor \frac{\log\left(\frac{102\cdot q_{32}}{\epsilon}\right)}{\log B}\right\rfloor + 1, M\right) = \left[26, 5.87\cdot 10^{12}\right).$$

Therefore, it remains to check equation (2) for $3 \leq m \leq 25$. For this, we use a program written in Mathematica and see that there are no solutions of the equation

$$L_n L_{n+1} L_{n+2} = \frac{6}{9}\left(10^m - 1\right).$$

This completes the proof of Theorem 2. □

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