An inequality involving a ratio of zeta functions

József Sándor

Department of Mathematics, Babeș-Bolyai University
Str. Kogălniceanu nr.1, 400084 Cluj, Romania
email: jsandor@math.ubbcluj.ro

Received: 7 October 2017  Accepted: 30 August 2018

Abstract: We prove an inequality for a ratio of zeta functions. This extends a classical result (see [2]). The method is based on Dirichlet series, combined with real analysis.

Keywords: Riemann zeta function, Dirichlet series, Inequalities for real functions.

2010 Mathematics Subject Classification: 11A25, 11N37, 26D20.

Let \( \omega(n) \) denote the number of distinct prime divisors of \( n \). Then \( \omega(1) = 0 \) and \( \omega(n) \) is an additive function, i.e.

\[
\omega(mn) = \omega(m) + \omega(n) \quad \text{for all} \quad (m, n) = 1.
\]

This implies immediately that the function

\[
f(n) = k^{\omega(n)}
\]

(where \( k \geq 2 \) is fixed) is a multiplicative function, i.e. satisfies the functional equation

\[
f(mn) = f(m) \cdot f(n) \quad \text{for all} \quad (m, n) = 1,
\]

where \( f(1) = 1 \).

A general Dirichlet series is an infinite series of type \( \sum_{n=1}^{\infty} \frac{a_n}{n^s} \), where \( s \in \mathbb{C} \) is such that the series is convergent. For \( a_n = 1 \), we get the Riemann zeta function

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
\]
which is convergent e.g. for all $\Re s > 1$. Another Dirichlet series is obtained when

$$a_n = f(n) = k^{\omega(n)}.$$

We will prove in what follows the following result:

**Theorem.** Let $s > 1$ a fixed positive integer. Then one has the inequality

$$\sum_{k=1}^{\infty} \frac{k^{\omega(n)}}{n^s} \leq \frac{\zeta^k(s)}{\zeta(k s)},$$

with equality only for $k = 2$.

For the proof, the following well-known result will be applied (see e.g. [1]).

**Lemma 1.** Let $f$ be a multiplicative arithmetical function, and let the series $\sum_{n=1}^{\infty} f(n)$ be absolutely convergent. Then we have the identity:

$$\sum_{n=1}^{\infty} f(n) = \prod_{p \text{ prime}} (1 + f(p) + f(p^2) + \ldots).$$

We shall need also the following auxiliary result:

**Lemma 2.** Let $0 < x \leq \frac{1}{2}$ and $k \geq 2$. Then

$$1 - x^k \geq (1 - x)^{k-1}[1 + x(k - 1)].$$

Inequality (4) may be written also as

$$1 - x^k \geq (1 - x)^{k-1}[(1 - x + kx)] = (1 - x)^k + kx(1 - x)^{k-1}.$$  

Let us define

$$g(x) = x^k = (1 - x)^k + kx(1 - x)^{k-1}, \; g : [0, 1] \to \mathbb{R}.$$  

We have to prove that $g(x) \leq 1$. One has

$$g(1) = g(0) = 1 \; \text{ and } \; g'(x) = kx[x^{k-2} - (k - 1)(1 - x)^{k-2}].$$

Remark that, as $0 < x \leq \frac{1}{2}$, we have $0 < x \leq 1 - x$, so

$$x^{k-2} \leq (1 - x)^{k-2} \leq (k - 1)(1 - x)^{k-2},$$

with equality only for $k = 2$. Thus we get $g'(x) \leq 0$, implying

$$g(x) \leq g(0) = 1.$$

**Remark.** The above proof shows that there is equality in (4) only for $k = 2$.

**Proof of Theorem.** Letting

$$f(n) = \frac{k^{\omega(n)}}{n^s}$$

in Lemma 1, we get
For \( f(n) = \frac{1}{n^s} \) in the same Lemma 1, we get Euler’s identity

\[
\sum_{k=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left( 1 + \frac{k}{p^s} + \frac{k}{p^{2s}} + \ldots \right) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}.
\]

Thus, by using Euler’s identity, we get

\[
\zeta(k s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^{ks}}},
\]

i.e.,

\[
\frac{\zeta^k(s)}{\zeta(k s)} = \prod_{p \text{ prime}} \frac{1 - \frac{1}{p^{ks}}}{\left( 1 - \frac{1}{p^s} \right)^k}.
\]

Put now \( x = \frac{1}{p^s} \). As \( s > 1 \) and \( p \geq 2 \), clearly \( x < \frac{1}{2} \). So we can apply Lemma 2, which implies

\[
\frac{1 - x^k}{(1 - x)^k} \geq \frac{1 + x(k - 1)}{1 - x}.
\]

In relation (5) one has

\[
1 + \frac{k}{p^s} + \frac{k}{p^{2s}} + \ldots = 1 + kx + kx^2 + \ldots = 1 + kx(1 + x + x^2 + \ldots) = 1 + \frac{kx}{x - 1} = \frac{1 + x(k - 1)}{x - 1}.
\]

By relations (6) and (7), this implies inequality (2), finishing the proof of Theorem. \( \square \)

**Remark.** For \( k = 2 \) we get the known identity (see [2])

\[
\sum_{k=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}.
\]

**References**
