One more disproof for the Legendre’s conjecture regarding the prime counting function $\pi(x)$

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Abstract: Let $\pi(x)$ denote the prime counting function, i.e., the number of primes not exceeding $x$. The Legendre’s conjecture regarding the prime counting function states that

$$\pi(x) = \frac{x}{\log x - A(x)},$$

where Legendre conjectured that $\lim_{x \to \infty} A(x) = 1.08366\ldots$, which is the Legendre’s constant. It is well-known that $\lim_{x \to \infty} A(x) = 1$, and hence the Legendre’s conjecture is not true. In this article we give various proofs of this limit and establish some generalizations.

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1 Introduction

The Prime Number Theorem (PNT) was first conjectured by the French mathematician Adrien-Marie Legendre (1752–1833) as an experiential statement. In 1798, Legendre asserted that $\pi(x) = x/(A \log x - B)$ for constants $A$ and $B$ [6]. A decade latter, in 1808 [7] he refined his
conjecture and formulated it by the form of

$$\pi(x) = \frac{x}{\log x - A(x)},$$

(1)

where Legendre conjectured that $A(x)$ is a function of $x$ such that

$$\lim_{x \to \infty} A(x) = 1.08366...,$$

which is the Legendre’s constant.

In 1848, Chebyshev showed that if the function $A(x)$ tended to a limit as $x \to \infty$, then necessarily the limit had to be 1 [1]. In 2017, R. Farhadian obtained the numerical values of the function $A(x)$ at points $10^k$ for $k = 1, 2, ..., 25$, and then, based on the numerical observations, he noted that the function $A(x)$ decreases to 1 (not necessarily monotonically) [5]. In this article we give various proofs of the limit

$$\lim_{x \to \infty} A(x) = 1,$$

proved by Vallée Poussin in 1899, and establish some generalizations.

## 2 Main results

By the Legendre’s formula (1) we have

$$A(x) = \frac{x}{\pi(x)} - \frac{x}{\log x}.$$  

(2)

We need the following lemmas.

**Lemma 2.1.**

$$\frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{c_1}{\log^2 x}\right) \leq \pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{c_2}{\log^2 x}\right),$$

where $c_1 = 1.8$ for $x \geq 32299$ and $c_2 = 2.51$ for $x \geq 355991$.

**Proof.** See [3] and [4].

**Lemma 2.2.** If $k$ is a finite real number, then

$$\lim_{x \to \infty} \frac{\log^3 x + k \log^2 x}{\log^3 x + \log^2 x + k \log x} = 1.$$  

**Proof.** We have

$$\frac{\log^3 x + k \log^2 x}{\log^3 x + \log^2 x + k \log x} = 1 + \frac{k}{\log x} + \frac{k}{\log^2 x}.$$  

□
Theorem 2.3. Given the function $A(x)$ in relation (2). Then

$$\lim_{x \to \infty} A(x) = 1.$$  \hfill (3)

Besides, we have the inequality

$$\frac{\log^3 x + 1.8 \log^2 x}{\log^3 x + \log^2 x + 1.8 \log x} \leq A(x) \leq \frac{\log^3 x + 2.51 \log^2 x}{\log^3 x + \log^2 x + 2.51 \log x},$$  \hfill (4)

where the inequality on the left hand holds for $x \geq 32299$ and the inequality on the right hand holds for $x \geq 355991$.

Proof. By Lemma 1 we know that

$$\frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) \leq \pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right).$$

By inverting the above inequality and multiplying the inequality by $x$, we have

$$\frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2.51 x}{\log^3 x} \leq \frac{x}{\pi(x)} \leq \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{1.8 x}{\log^3 x},$$

simplifying,

$$\frac{\log x}{1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}} \leq \frac{x}{\pi(x)} \leq \frac{\log x}{1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}}.$$  \hfill (5)

Clearly, if $\log x$ minus each part of the inequality (5), then we have (see equation (2))

$$\log x - \frac{\log x}{1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}} \leq A(x) \leq \log x - \frac{\log x}{1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}}.$$  

Consequently

$$\frac{\log^3 x + 1.8 \log^2 x}{\log^3 x + \log^2 x + 1.8 \log x} \leq A(x) \leq \frac{\log^3 x + 2.51 \log^2 x}{\log^3 x + \log^2 x + 2.51 \log x},$$

that is, inequality (4). Finally, inequality (4) and Lemma 2.2 give limit (3).

In the following theorem we obtain an asymptotic formula for the function $A(x)$. An immediate corollary of this asymptotic formula is the limit $\lim_{x \to \infty} A(x) = 1$.

Theorem 2.4. Let $h \geq 3$ be an arbitrary but fixed positive integer. The following asymptotic formula holds

$$A(x) = \frac{1 + \sum_{k=1}^{h-2} \frac{(k+1)!}{\log^k x} + o \left( \frac{1}{\log^{h-2} x} \right)}{1 + \sum_{k=1}^{h-1} \frac{k!}{\log^k x} + o \left( \frac{1}{\log^{h-1} x} \right)},$$

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Proof. The following asymptotic formula for the prime counting function $\pi(x)$ is well-known \[8\]

$$\pi(x) = \sum_{k=1}^{h} \frac{(k-1)!x}{\log^k x} + o\left(\frac{x}{\log^h x}\right), \tag{6}$$

where $h$ is an arbitrary but fixed positive integer.

Therefore we have

$$\frac{x}{\pi(x)} = \frac{\log x}{\sum_{k=1}^{h} \frac{(k-1)!}{\log^{k-1} x} + o\left(\frac{1}{\log^{h-1} x}\right)},$$

and consequently

$$A(x) = \log x - \frac{x}{\pi(x)} = \log x \left(1 - \frac{1}{\sum_{k=1}^{h} \frac{(k-1)!}{\log^{k-1} x} + o\left(\frac{1}{\log^{h-1} x}\right)}\right)$$

$$= \log x \frac{\sum_{k=2}^{h} \frac{(k-1)!}{\log^{k-1} x} + o\left(\frac{1}{\log^{h-1} x}\right)}{1 + \sum_{k=2}^{h} \frac{(k-1)!}{\log^{k-1} x} + o\left(\frac{1}{\log^{h-1} x}\right)}$$

$$= 1 + \sum_{k=3}^{h} \frac{(k-1)!}{\log^{k-2} x} + o\left(\frac{1}{\log^{h-2} x}\right) = 1 + \sum_{k=1}^{h-2} \frac{(k+1)!}{\log^{k+1} x} + o\left(\frac{1}{\log^{h-2} x}\right).$$

Now, we give another proof that $\lim_{x \to \infty} A(x) = 1$ using formula for the $n$-th prime $p_n$ and $\log p_n$. We put $d_n = p_{n+1} - p_n$.

Theorem 2.5. The following limit holds

$$\lim_{x \to \infty} A(x) = 1.$$  \[\Box\]

Proof. The prime number theorem $p_n \sim n \log n$ implies the following formula for $\log p_n$

$$\log p_n = \log n + \log \log n + o(1). \tag{7}$$

On the other hand, a consequence of the equation (see (6) with $h = 2$)

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + o\left(\frac{x}{\log^2 x}\right)$$

is the following well-known formula for $p_n$ \[2\]

$$p_n = n \log n + n \log \log n - n + o(n). \tag{8}$$

Therefore, we have (by (7) and (8))

$$A(p_n) = \log p_n - \frac{p_n}{\pi(p_n)} = \log p_n - \frac{p_n}{n} = 1 + o(1), \tag{9}$$

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and consequently

\[ A(p_{n+1}) = \log p_{n+1} - \frac{p_{n+1}}{\pi(p_{n+1})} = \log p_{n+1} - \frac{p_{n+1}}{n+1} = 1 + o(1). \tag{10} \]

Now, we have (by (9) and (10))

\[
A(p_n) = A(p_{n+1} - d_n) = \log(p_{n+1} - d_n) - \frac{p_{n+1} - d_n}{n} = \log p_{n+1} + \log \left(1 - \frac{d_n}{p_{n+1}}\right) - \frac{p_{n+1}}{n(n+1)} + \frac{d_n}{n} = 1 + o(1) + \frac{d_n}{n} = 1 + o(1).
\]

Therefore

\[ \frac{d_n}{n} = o(1). \]

Let us consider an arbitrary sequence \( a_n \) such that \( \lim_{n \to \infty} a_n = \infty \). For each \( n \) there is a prime \( p_{n'} \) such that \( p_{n'} < a_n \leq p_{n'+1} \). Hence \( a_n = p_{n'+1} - b_{n'} \), where \( 0 \leq b_{n'} < d_{n'} \). If in the sequence \( a_n \) there is a subsequence of prime numbers \( p_{n'+1} \), we have (see above) \( A(p_{n'+1}) = 1 + o(1) \), therefore, we consider the subsequence of \( a_n \) such that \( a_n \neq p_{n'+1} \) and consequently \( \pi(a_n) = n' \).

For this subsequence we have

\[ A(a_n) = A(p_{n'+1} - b_{n'}) = \log(p_{n'+1} - b_{n'}) - \frac{p_{n'+1} - b_{n'}}{n'} = \cdots = 1 + o(1). \]

Therefore for the complete sequence \( a_n \) we have \( A(a_n) = 1 + o(1) \). Consequently, by a well-known theorem of analysis [9], we have \( \lim_{x \to \infty} A(x) = 1 \).

Let \( k \) be an arbitrary but fixed positive integer and let us consider the sequence \( p_n^k \), that is, the sequence of the \( k \)-th powers of the prime numbers. In particular, if \( k = 1 \), we obtain the sequence of primes; if \( k = 2 \), we obtain the sequence of squares of primes, etc.

The Prime Number Theorem establishes

\[ \pi(x) \sim \frac{x}{\log x}. \]

Let \( \pi_k(x) \) be the number of \( k \)-th powers of primes not exceeding \( x \), that is, \( p_n^k \leq x \), then \( \pi_1(x) = \pi(x) \). The Prime Number Theorem gives

\[ \pi_k(x) = \pi(\sqrt[k]{x}) \sim \frac{\sqrt[k]{x}}{\log \sqrt[k]{x}}. \]

We have the following generalization of the Legendre’s formula.

**Theorem 2.6.** Let \( s \) and \( k \) be arbitrary but fixed positive integers. We have

\[ (\pi_k(x))^s = \frac{(\sqrt[k]{x})^s}{\log^s \sqrt[k]{x} - A_{k,s}(x) \log^{s-1} \sqrt[k]{x}}, \]

where \( \lim_{x \to \infty} A_{k,s}(x) = s \).
Proof. We have (see (6) with \( h = 2 \))

\[
\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + o \left( \frac{x}{\log^2 x} \right).
\]

Therefore

\[
\pi_k(x) = \pi \left( \sqrt[k]{x} \right) = \frac{\sqrt[k]{x}}{\log \sqrt[k]{x}} + \frac{\sqrt[k]{x}}{\log^2 \sqrt[k]{x}} + o \left( \frac{\sqrt[k]{x}}{\log^2 \sqrt[k]{x}} \right),
\]

and consequently

\[
\left( \pi_k(x) \right)^s = \left( \frac{\sqrt[k]{x}}{\log^s \sqrt[k]{x}} \right)^s \left( \pi_k(x) \right)^s = \left( \frac{\sqrt[k]{x}}{\log^s \sqrt[k]{x}} \right)^s \left( 1 + \frac{1}{\log \sqrt[k]{x}} + o \left( \frac{1}{\log \sqrt[k]{x}} \right) \right)^s
\]

Now, we have

\[
A_{k,s}(x) = \log \sqrt[k]{x} - \frac{1}{\log^{s-1} \sqrt[k]{x}} \left( \pi_k(x) \right)^s
\]

\[
= \log \sqrt[k]{x} - \frac{\log \sqrt[k]{x}}{1 + \frac{s}{\log \sqrt[k]{x}} + o \left( \frac{1}{\log \sqrt[k]{x}} \right)} = \log \sqrt[k]{x} \frac{s}{\log \sqrt[k]{x}} + o \left( \frac{1}{\log \sqrt[k]{x}} \right)
\]

\[
= \frac{s + o(1)}{1 + o(1)} = s + o(1).
\]

\[\square\]

Corollary 2.7. We have

\[
\pi_k(x) = \frac{\sqrt[k]{x}}{\log \sqrt[k]{x} - A_{k,1}(x)},
\]

where \( \lim_{x \to \infty} A_{k,1}(x) = 1 \). Besides

\[
\lim_{x \to \infty} A(x) = 1.
\]

Proof. It is the case \( s = 1 \) in the former theorem. Note that the limit 1 does not depend of \( k \). On the other hand, \( A(x) = A_{1,1}(x) \).

In the following theorem we establish other generalization of the Legendre’s formula. We have the asymptotic formula (see (6))

\[
\pi(x) = \sum_{k=1}^{h} \frac{(k-1)!x}{\log^k x} + o \left( \frac{x}{\log^h x} \right).
\]

If \( h = 1 \), we obtain the Prime Number Theorem

\[
\pi(x) = \frac{x}{\log x} + o \left( \frac{x}{\log x} \right).
\]
The Legendre’s formula is
\[ \pi(x) = \frac{x}{\log x - A(x)}, \]
where \( \lim_{x \to \infty} A(x) = 1. \)
If \( h = 2 \), we obtain
\[ \pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + o\left(\frac{x}{\log^2 x}\right). \]
If we put the formula
\[ \pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x - A_2(x) \log x}, \]
then we shall prove in the following theorem that \( \lim_{x \to \infty} A_2(x) = 2 \), etc.

In general, we have the following theorem.

**Theorem 2.8.** Let \( h \geq 2 \) be an arbitrary but fixed positive integer. We have the following formula
\[ \pi(x) = \sum_{k=1}^{h-1} \frac{(k-1)!x}{\log^k x} + \frac{(h-1)!x}{\log^h x - A_h(x) \log^{h-1} x}, \]
where
\[ \lim_{x \to \infty} A_h(x) = h. \]

**Proof:** We have
\[ A_h(x) = \log x - \frac{1}{\log^{h-1} x} \pi(x) - \sum_{k=1}^{h-1} \frac{(k-1)!x}{\log^k x}. \]
On the other hand, we have the equation
\[ \pi(x) = \sum_{k=1}^{h+2} \frac{(k-1)!x}{\log^k x} + o\left(\frac{x}{\log^{h+2} x}\right). \]
Substituting this equation in the former equation we find that
\[
A_h(x) = \log x - \frac{1}{\log^{h-1} x} \frac{(h-1)!x}{\log^h x} - \frac{h!x}{\log^{h+1} x} - \frac{(h+1)!x}{\log^{h+2} x} + o\left(\frac{x}{\log^{h+2} x}\right) \\
= \log x \left(1 - \frac{(h-1)!}{(h-1)! + \frac{h!}{\log x} + \frac{(h+1)!}{\log^2 x} + o\left(\frac{1}{\log^2 x}\right)}\right) \\
= \frac{h! + \frac{(h+1)!}{\log x} + o\left(\frac{1}{\log x}\right)}{(h-1)! + \frac{h!}{\log x} + \frac{(h+1)!}{\log^2 x} + o\left(\frac{1}{\log^2 x}\right)} = h + o(1). \]

\[ \square \]
Cipolla’s asymptotic formula for the \( n \)-th prime \( p_n \) is [2]

\[
p_n = n \log n + n \log \log n - n + \sum_{i=1}^{r} \frac{(-1)^{i-1}nP_i(\log \log n)}{i! \log^i n} + o \left( \frac{n}{\log^r n} \right),
\]

where \( r \) is an arbitrary but fixed positive integer and \( P_i(x) \) is a polynomial of degree \( i \) and leading coefficient \((i - 1)!\).

**Theorem 2.9.** The following asymptotic formula holds

\[
p_n = n \log n + n \log \log n - n + \sum_{i=1}^{r-1} \frac{(-1)^{i-1}nP_i(\log \log n)}{i! \log^i n} + \frac{nP_r(\log \log n)}{r! \log^r n} + B_r(n)\frac{r! \log^r \log n}{r+1},
\]

where \( \lim_{n \to \infty} B_r(n) = \frac{r}{r+1} \).

**Proof.** The proof is similar to the proof of the former theorem. \(\square\)

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**References**


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