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One more disproof for the Legendre's conjecture regarding the prime counting function $\pi(x)$

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Abstract: Let $\pi(x)$ denote the prime counting function, i.e., the number of primes not exceeding x. The Legendre's conjecture regarding the prime counting function states that

$$\pi(x) = \frac{x}{\log x - A(x)},$$

where Legendre conjectured that $\lim_{x\to\infty} A(x) = 1.08366...$, which is the Legendre's constant. It is well-known that $\lim_{x\to\infty} A(x) = 1$, and hence the Legendre's conjecture is not true. In this article we give various proofs of this limit and establish some generalizations.

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1 Introduction

The Prime Number Theorem (PNT) was first conjectured by the French mathematician Adrien-Marie Legendre (1752–1833) as an experiential statement. In 1798, Legendre asserted that $\pi(x) = x/(A\log x - B)$ for constants A and B [6]. A decade latter, in 1808 [7] he refined his conjecture and formulated it by the form of

$$\pi(x) = \frac{x}{\log x - A(x)}, \tag{1}$$

where Legendre conjectured that $A(x)$ is a function of x such that

$$\lim_{x\to\infty} A(x) = 1.08366...,$$

which is the Legendre's constant.

In 1848, Chebyshev showed that if the function $A(x)$ tended to a limit as $x \to \infty$, then necessarily the limit had to be 1 [1]. In 2017, R. Farhadian obtained the numerical values of the function $A(x)$ at points 10^k for $k = 1, 2, ..., 25$, and then, based on the numerical observations, he noted that the function $A(x)$ decreases to 1 (not necessarily monotonically) [5]. In this article we give various proofs of the limit

$$\lim_{x\to\infty} A(x) = 1,$$

proved by Vallée Poussin in 1899, and establish some generalizations.

2 Main results

By the Legendre's formula (1) we have

$$A(x) = \log x - \frac{x}{\pi(x)}. \tag{2}$$

We need the following lemmas.

Lemma 2.1.

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{c_1}{\log^2 x}\right) \le \pi(x) \le \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{c_2}{\log^2 x}\right),$$

where $c_1 = 1.8$ *for* $x \ge 32299$ *and* $c_2 = 2.51$ *for* $x \ge 355991$.

Proof. See [3] and [4]. □

Lemma 2.2. *If* k *is a finite real number, then*

$$\lim_{x\to\infty} \frac{\log^3 x + k\log^2 x}{\log^3 x + \log^2 x + k\log x} = 1.$$

Proof. We have

$$\frac{\log^3 x + k\log^2 x}{\log^3 x + \log^2 x + k\log x} = \frac{1 + \frac{k}{\log x}}{1 + \frac{1}{\log x} + \frac{k}{\log^2 x}}.$$

□

Theorem 2.3. *Given the function $A(x)$ in relation (2). Then*

$$\lim_{x\to\infty} A(x) = 1. \tag{3}$$

Besides, we have the inequality

$$\frac{\log^3 x + 1.8\log^2 x}{\log^3 x + \log^2 x + 1.8\log x} \le A(x) \le \frac{\log^3 x + 2.51\log^2 x}{\log^3 x + \log^2 x + 2.51\log x}, \tag{4}$$

where the inequality on the left hand holds for $x \ge 32299$ and the inequality on the right hand holds for $x \ge 355991$.

Proof. By Lemma 1 we know that

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) \le \pi(x) \le \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right).$$

By inverting the above inequality and multiplying the inequality by x, we have

$$\frac{x}{\frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2.51x}{\log^3 x}} \le \frac{x}{\pi(x)} \le \frac{x}{\frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{1.8x}{\log^3 x}},$$

simplifying,

$$\frac{\log x}{1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}} \le \frac{x}{\pi(x)} \le \frac{\log x}{1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}}. \tag{5}$$

Clearly, if $\log x$ minus each part of the inequality (5), then we have (see equation (2))

$$\log x - \frac{\log x}{1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}} \le A(x) \le \log x - \frac{\log x}{1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}}.$$

Consequently

$$\frac{\log^3 x + 1.8\log^2 x}{\log^3 x + \log^2 x + 1.8\log x} \le A(x) \le \frac{\log^3 x + 2.51\log^2 x}{\log^3 x + \log^2 x + 2.51\log x},$$

that is, inequality (4). Finally, inequality (4) and Lemma 2.2 give limit (3). □

In the following theorem we obtain an asymptotic formula for the function $A(x)$. An immediate corollary of this asymptotic formula is the limit $\lim_{x\to\infty} A(x) = 1$.

Theorem 2.4. *Let $h \ge 3$ be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$A(x) = \frac{1 + \sum_{k=1}^{h-2} \frac{(k+1)!}{\log^k x} + o\left(\frac{1}{\log^{h-2} x}\right)}{1 + \sum_{k=1}^{h-1} \frac{k!}{\log^k x} + o\left(\frac{1}{\log^{h-1} x}\right)}.$$

Proof. The following asymptotic formula for the prime counting function $\pi(x)$ is well-known [8]

$$\pi(x) = \sum_{k=1}^{h} \frac{(k-1)!x}{\log^k x} + o\left(\frac{x}{\log^h x}\right), \tag{6}$$

where h is an arbitrary but fixed positive integer.

Therefore we have

$$\frac{x}{\pi(x)} = \frac{\log x}{\sum_{k=1}^{h} \frac{(k-1)!}{\log^{k-1} x} + o\left(\frac{1}{\log^{h-1} x}\right)},$$

and consequently

$$\begin{aligned} A(x) &= \log x - \frac{x}{\pi(x)} = \log x \left(1 - \frac{1}{\sum_{k=1}^{h} \frac{(k-1)!}{\log^{k-1} x} + o\left(\frac{1}{\log^{h-1} x}\right)}\right) \\ &= \log x \frac{\sum_{k=2}^{h} \frac{(k-1)!}{\log^{k-1} x} + o\left(\frac{1}{\log^{h-1} x}\right)}{1 + \sum_{k=2}^{h} \frac{(k-1)!}{\log^{k-1} x} + o\left(\frac{1}{\log^{h-1} x}\right)} \\ &= \frac{1 + \sum_{k=3}^{h} \frac{(k-1)!}{\log^{k-2} x} + o\left(\frac{1}{\log^{h-2} x}\right)}{1 + \sum_{k=2}^{h} \frac{(k-1)!}{\log^{k-1} x} + o\left(\frac{1}{\log^{h-1} x}\right)} = \frac{1 + \sum_{k=1}^{h-2} \frac{(k+1)!}{\log^{k} x} + o\left(\frac{1}{\log^{h-2} x}\right)}{1 + \sum_{k=1}^{h-1} \frac{k!}{\log^{k} x} + o\left(\frac{1}{\log^{h-1} x}\right)}. \end{aligned}$$

□

Now, we give other proof that $\lim_{x\to\infty} A(x) = 1$ using formula for the n-th prime p_n and $\log p_n$. We put $d_n = p_{n+1} - p_n$.

Theorem 2.5. *The following limit holds*

$$\lim_{x\to\infty} A(x) = 1.$$

Proof. The prime number theorem $p_n \sim n \log n$ implies the following formula for $\log p_n$

$$\log p_n = \log n + \log\log n + o(1). \tag{7}$$

On the other hand, a consequence of the equation (see (6) with $h = 2$)

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + o\left(\frac{x}{\log^2 x}\right)$$

is the following well-known formula for p_n [2]

$$p_n = n \log n + n \log\log n - n + o(n). \tag{8}$$

Therefore, we have (by (7) and (8))

$$A(p_n) = \log p_n - \frac{p_n}{\pi(p_n)} = \log p_n - \frac{p_n}{n} = 1 + o(1), \tag{9}$$

and consequently

$$A(p_{n+1}) = \log p_{n+1} - \frac{p_{n+1}}{\pi(p_{n+1})} = \log p_{n+1} - \frac{p_{n+1}}{n+1} = 1 + o(1). \tag{10}$$

Now, we have (by (9) and (10))

$$A(p_n) = A(p_{n+1} - d_n) = \log(p_{n+1} - d_n) - \frac{p_{n+1} - d_n}{n} = \log p_{n+1}$$

$$+ \ \log\left(1 - \frac{d_n}{p_{n+1}}\right) - \frac{p_{n+1}}{n+1} - \frac{p_{n+1}}{n(n+1)} + \frac{d_n}{n} = 1 + o(1) + \frac{d_n}{n} = 1 + o(1).$$

Therefore

$$\frac{d_n}{n} = o(1).$$

Let us consider an arbitrary sequence a_n such that $\lim_{n\to\infty} a_n = \infty$. For each n there is a prime $p_{n'}$ such that $p_{n'} < a_n \le p_{n'+1}$. Hence $a_n = p_{n'+1} - b_{n'}$, where $0 \le b_{n'} < d_{n'}$. If in the sequence a_n there is a subsequence of prime numbers $p_{n'+1}$, we have (see above) $A(p_{n'+1}) = 1 + o(1)$, therefore, we consider the subsequence of a_n such that $a_n \ne p_{n'+1}$ and consequently $\pi(a_n) = n'$. For this subsequence we have

$$A(a_n) = A(p_{n'+1} - b_{n'}) = \log(p_{n'+1} - b_{n'}) - \frac{p_{n'+1} - b_{n'}}{n'} = \cdots = 1 + o(1).$$

Therefore for the complete sequence a_n we have $A(a_n) = 1 + o(1)$. Consequently, by a well-known theorem of analysis [9], we have $\lim_{x\to\infty} A(x) = 1$. □

Let k be an arbitrary but fixed positive integer and let us consider the sequence p_n^k, that is, the sequence of the k-th powers of the prime numbers. In particular, if $k = 1$, we obtain the sequence of primes; if $k = 2$, we obtain the sequence of squares of primes, etc.

The Prime Number Theorem establishes

$$\pi(x) \sim \frac{x}{\log x}.$$

Let $\pi_k(x)$ be the number of k-th powers of primes not exceeding x, that is, $p_n^k \le x$, then $\pi_1(x) = \pi(x)$. The Prime Number Theorem gives

$$\pi_k(x) = \pi\left(\sqrt[k]{x}\right) \sim \frac{\sqrt[k]{x}}{\log \sqrt[k]{x}}.$$

We have the following generalization of the Legendre's formula.

Theorem 2.6. *Let s and k be arbitrary but fixed positive integers. We have*

$$(\pi_k(x))^s = \frac{\left(\sqrt[k]{x}\right)^s}{\log^s \sqrt[k]{x} - A_{k,s}(x)\log^{s-1}\sqrt[k]{x}},$$

where $\lim_{x\to\infty} A_{k,s}(x) = s$.

Proof. We have (see (6) with $h = 2$)

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + o\left(\frac{x}{\log^2 x}\right).$$

Therefore

$$\pi_k(x) = \pi\left(\sqrt[k]{x}\right) = \frac{\sqrt[k]{x}}{\log \sqrt[k]{x}} + \frac{\sqrt[k]{x}}{\log^2 \sqrt[k]{x}} + o\left(\frac{\sqrt[k]{x}}{\log^2 \sqrt[k]{x}}\right),$$

and consequently

$$\begin{aligned}(\pi_k(x))^s &= \frac{\left(\sqrt[k]{x}\right)^s}{\log^s \sqrt[k]{x}}\left(1 + \frac{1}{\log \sqrt[k]{x}} + o\left(\frac{1}{\log \sqrt[k]{x}}\right)\right)^s \\ &= \frac{\left(\sqrt[k]{x}\right)^s}{\log^s \sqrt[k]{x}}\left(1 + \frac{s}{\log \sqrt[k]{x}} + o\left(\frac{1}{\log \sqrt[k]{x}}\right)\right).\end{aligned}$$

Now, we have

$$\begin{aligned}&A_{k,s}(x) = \log \sqrt[k]{x} - \frac{1}{\log^{s-1} \sqrt[k]{x}} \frac{\left(\sqrt[k]{x}\right)^s}{(\pi_k(x))^s} \\ =\ & \log \sqrt[k]{x} - \frac{\log \sqrt[k]{x}}{1 + \frac{s}{\log \sqrt[k]{x}} + o\left(\frac{1}{\log \sqrt[k]{x}}\right)} = \log \sqrt[k]{x} \frac{\frac{s}{\log \sqrt[k]{x}} + o\left(\frac{1}{\log \sqrt[k]{x}}\right)}{1 + \frac{s}{\log \sqrt[k]{x}} + o\left(\frac{1}{\log \sqrt[k]{x}}\right)} \\ =\ & \frac{s + o(1)}{1 + o(1)} = s + o(1).\end{aligned}$$

□

Corollary 2.7. *We have*

$$\pi_k(x) = \frac{\sqrt[k]{x}}{\log \sqrt[k]{x} - A_{k,1}(x)},$$

where $\lim_{x\to\infty} A_{k,1}(x) = 1$. *Besides*

$$\lim_{x\to\infty} A(x) = 1.$$

Proof. It is the case $s = 1$ in the former theorem. Note that the limit 1 does not depend of k. On the other hand, $A(x) = A_{1,1}(x)$. □

In the following theorem we establish other generalization of the Legendre's formula. We have the asymptotic formula (see (6))

$$\pi(x) = \sum_{k=1}^{h} \frac{(k-1)!x}{\log^k x} + o\left(\frac{x}{\log^h x}\right).$$

If $h = 1$, we obtain the Prime Number Theorem

$$\pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right).$$

The Legendre's formula is

$$\pi(x) = \frac{x}{\log x - A(x)},$$

where $\lim_{x\to\infty} A(x) = 1$.
If $h = 2$, we obtain

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + o\left(\frac{x}{\log^2 x}\right).$$

If we put the formula

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x - A_2(x)\log x},$$

then we shall prove in the following theorem that $\lim_{x\to\infty} A_2(x) = 2$, etc.

In general, we have the following theorem.

Theorem 2.8. *Let $h \geq 2$ be an arbitrary but fixed positive integer. We have the following formula*

$$\pi(x) = \sum_{k=1}^{h-1} \frac{(k-1)!x}{\log^k x} + \frac{(h-1)!x}{\log^h x - A_h(x)\log^{h-1} x},$$

where

$$\lim_{x\to\infty} A_h(x) = h.$$

Proof: We have

$$A_h(x) = \log x - \frac{1}{\log^{h-1} x} \frac{(h-1)!x}{\pi(x) - \sum_{k=1}^{h-1} \frac{(k-1)!x}{\log^k x}}.$$

On the other hand, we have the equation

$$\pi(x) = \sum_{k=1}^{h+2} \frac{(k-1)!x}{\log^k x} + o\left(\frac{x}{\log^{h+2} x}\right).$$

Substituting this equation in the former equation we find that

$$\begin{aligned} A_h(x) &= \log x - \frac{1}{\log^{h-1} x} \frac{(h-1)!x}{\frac{(h-1)!x}{\log^h x} + \frac{h!x}{\log^{h+1} x} + \frac{(h+1)!x}{\log^{h+2} x} + o\left(\frac{x}{\log^{h+2} x}\right)} \\ &= \log x \left(1 - \frac{(h-1)!}{(h-1)! + \frac{h!}{\log x} + \frac{(h+1)!}{\log^2 x} + o\left(\frac{1}{\log^2 x}\right)}\right) \\ &= \frac{h! + \frac{(h+1)!}{\log x} + o\left(\frac{1}{\log x}\right)}{(h-1)! + \frac{h!}{\log x} + \frac{(h+1)!}{\log^2 x} + o\left(\frac{1}{\log^2 x}\right)} = h + o(1). \end{aligned}$$

□

Cipolla's asymptotic formula for the n-th prime p_n is [2]

$$p_n = n \log n + n \log \log n - n + \sum_{i=1}^{r} \frac{(-1)^{i-1} n P_i(\log \log n)}{i! \log^i n} + o\left(\frac{n}{\log^r n}\right),$$

where r is an arbitrary but fixed positive integer and $P_i(x)$ is a polynomial of degree i and leading coefficient $(i-1)!$.

Theorem 2.9. *The following asymptotic formula holds*

$$\begin{aligned} p_n &= n \log n + n \log \log n - n + \sum_{i=1}^{r-1} \frac{(-1)^{i-1} n P_i(\log \log n)}{i! \log^i n} \\ &+ (-1)^{r-1} \frac{n P_r(\log \log n)}{r! \log^r n + B_r(n) r! \log^{r-1} n \log \log n}, \end{aligned}$$

where $\lim_{n\to\infty} B_r(n) = \frac{r}{r+1}$.

Proof. The proof is similar to the proof of the former theorem. □

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