On products of quartic polynomials over consecutive indices which are perfect squares

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Abstract: Let \( a \) be a positive integer. We study the Diophantine equation
\[
\prod_{k=1}^{n} (a^2 k^4 + (2a - a^2) k^2 + 1) = y^2.
\]
This Diophantine equation generalizes a result of Gürel [5] for \( a = 2 \). We also prove that the product \((2^2 - 1)(3^2 - 1) \ldots (n^2 - 1)\) is a perfect square only for the values \( n \) for which the triangular number \( T_n \) is a perfect square.

Keywords: Diophantine equation, Perfect square, Quartic polynomial, Quadratic polynomial.

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1 Introduction

The study of sequences containing infinitely many squares is a common topic in number theory. Let \( \Omega_\mu(n) = (1^\mu + 1)(2^\mu + 1) \ldots (n^\mu + 1) \) where \( \mu \geq 2 \) is an integer. Amdeberhan et al. [1] conjectured that \( \Omega_3(n) \) is not a square for any integer \( n > 3 \). Cilleruelo [3] confirmed this conjecture. Gürel and Kisisel [6] proved that \( \Omega_3(n) \) is not a square. Later, an idea due to Zudilin was applied to \( \Omega_\mu(n) \) by Zhang and Wang [9] and to \( \Omega_\mu(n) \) by Chen et al. [2] for any odd prime \( p \). Fang [4] confirmed another similar conjecture posed by Amdeberhan et al. [1] to the
products of quadratic polynomials $\prod_{k=1}^{n}(4k^2+1)$ and $\prod_{k=1}^{n}(2k^2-2k+1)$ are not perfect squares. Yang et al. [8] studied the Diophantine equation $\prod_{k=1}^{n}(ak^2 + bk + c) = dy^2$. Gürel [5] proved that the product $\prod_{k=1}^{n}(4k^4 + 1)$ is a perfect square for infinitely many $n$.

In this manuscript, we will extend the result of Gürel [5] on the polynomial $4k^4 + 1$ to the polynomial $P_a(k) = a^2k^4 + (2a - a^2)k^2 + 1$, where $a$ is a positive integer. Next, we prove the product $(2^2 - 1)(3^2 - 1)\ldots(n^2 - 1)$ is a perfect square only for the values $n$ for which the triangular number $T_n$ is a perfect square.

## 2 Main results

Let $a$ be a positive integer and $P_a(x) = a^2x^4 + (2a - a^2)x^2 + 1$. Denote $X_a(n)$ is the product of first $n$ consecutive values of the $P_a(n)$, i.e.,

$$X_a(n) = P_a(1)P_a(2)\ldots P_a(n).$$

**Lemma 1.** $X_a(n)$ is a square if and only if $an^2 + an + 1$ is a square.

**Proof.** Let $f(x) = ax^2 - ax + 1$. Then $f(x + 1) = ax^2 + ax + 1$ and

$$P_a(x) = a^2x^4 + (2a - a^2)x^2 + 1 = (ax^2 - ax + 1)(ax^2 + ax + 1) = f(x)f(x + 1).$$

We have

$$X_a(n) = \prod_{k=1}^{n} P_a(k) = \prod_{k=1}^{n} f(k)f(k + 1) = \left( \prod_{k=2}^{n} f(k) \right)^2 f(1)f(n + 1).$$

Since $f(1) = 1$, it follows that $X_a(n)$ is a square if and only if $f(n + 1)$ is a square. \qed

Consider $4n^2 + 4n + 1 = (2n + 1)^2$, we obtain that

$$\prod_{k=1}^{n}(16k^4 - 8k^2 + 1)$$

is a perfect square for all $n$.

**Theorem 1.** Let $a, d, n$ be positive integers with $a = d^2 \neq 4$. Suppose $p = \left\lfloor \frac{d + 1}{2} \right\rfloor$. Then $X_a(n)$ is not a perfect square for $n > \frac{p^2 - 2p}{d^2 - 2dp + 2d}$

**Proof.** By Lemma 1, the problem is reduced to finding square values of $f(n + 1)$, i.e., finding integer solutions to the following equation,

$$an^2 + an + 1 = m^2. \tag{1}$$

We see that

$$an^2 + an + 1 = d^2n^2 + d^2n + 1 < (dn + p)^2.$$

Assume $n > (p^2 - 2p)/(d^2 - 2dp + 2d)$. 57
If \( d \) is even, we get that \( p = \frac{d}{2} \) and \( n > \frac{d - 4}{8} \), so

\[
d^2 n^2 + d^2 n + 1 > (dn + p - 1)^2.
\]

And if \( d \) is odd, we have that \( p = (d + 1)/2 \) and \( n > (d^2 - 2d - 3)/4d \), so

\[
d^2 n^2 + d^2 n + 1 > (dn + p - 1)^2.
\]

We obtain that

\[
(dn + p - 1)^2 < d^2 n^2 + d^2 n + 1 < (dn + p)^2.
\]

Since there is no perfect square between two consecutive perfect squares, \( X_a(n) \) is not a perfect square for \( n > \frac{p^2 - 2p}{d^2 - 2dp + 2d} \) and \( a = d^2 \neq 4 \).

For \( a \) not a perfect square, we conjecture that the Diophantine equation (1) has infinitely many solutions. The case \( a = 2 \) has been shown in [5]. In the next theorem, we will only show the case \( 3 \leq a \leq 13 \).

**Theorem 2.** Let \( 3 \leq a \leq 13 \) be not a perfect square. Then \( X_a(n) \) is a perfect square for infinitely many \( n \).

**Proof.** It suffices to find the integer solutions of (1).

- Case \( a = 3 \), we consider

\[
3n^2 + 3n + 1 = m^2. 
\]

We see that \( (7, 13) \) is a solution of (2). For each solution \( (x, y) \) of (2), the map sends \( (x, y) \) to \((7x + 4y + 3, 12x + 7y + 6)\), which gives another solution of (2). It can be verified that

\[
3(7x + 4y + 3)^2 + 3(7x + 4y + 3) + 1 = 147x^2 + 48y^2 + 168xy + 147x + 84y + 37
\]

\[
= (3x^2 + 3x + 1) - y^2 + (12x + 7y + 6)^2
\]

\[
= (12x + 7y + 6)^2.
\]

Therefore, the equation (2) has infinitely many distinct solutions.

(Note: If \( (n_i, m_i) \) are all solutions of (2), the sequence \{\( n_i \)\} satisfies \( n_i = 14n_{i-1} - n_{i-2} + 6 \), where \( n_0 = 0, n_1 = 7 \), see A001921 in [7], and the sequence \{\( m_i \)\} is A001570 in [7].)

- Case \( a = 5 \), we consider

\[
5n^2 + 5n + 1 = m^2. 
\]

Clearly, \( (8, 19) \) is a solution of (3). For each solution \( (x, y) \) of (3), the map sends \( (x, y) \) to \((9x + 4y + 4, 20x + 9y + 10)\), which gives another solution of (3).

(Note: If \( (n_i, m_i) \) are all solutions of (3), then \( n_i = (F_{6i+3} - 2)/4 \) and \( m_i = (F_{6i+4} + F_{6i+2})/4 \), where \( F_i \) is \( i^{th} \) Fibonacci number; see A053606 and A049629 in [7].)
• Case $a = 6$, we consider

$$6n^2 + 6n + 1 = m^2.$$  \hspace{1cm} (4)

Clearly, $(4, 11)$ is a solution of $(4)$. For each solution $(x, y)$ of $(4)$, the map sends $(x, y)$ to
$(5x + 2y + 2, 12x + 5y + 6)$, which gives another solution of $(4)$.
(Note: If $(n_i, m_i)$ are all solutions of $(4)$, then $n_i = 11n_{i-1} - 11n_{i-2} + n_{i-3}$, where $n_0 = 0$, $n_1 = 4$ and $n_2 = 44$, see A105038 in [7] and $m_i$ is A054320 in [7].)

• Case $a = 7$, we consider

$$7n^2 + 7n + 1 = m^2.$$  \hspace{1cm} (5)

Clearly, $(15, 41)$ is a solution of $(5)$. For each solution $(x, y)$ of $(5)$, the map sends $(x, y)$ to
$(127x + 48y + 63, 336x + 127y + 168)$, which gives another solution of $(5)$.
(Note: If $(n_i, m_i)$ are all solutions of $(5)$, then $n_i = 254n_{i-2} - n_{i-4} + 126$, where $n_0 = 0$, $n_1 = 15$, $n_2 = 111$ and $n_3 = 3936$, see A105051 or A105040 in [7].)

• Case $a = 8$, we consider

$$8n^2 + 8n + 1 = m^2.$$  \hspace{1cm} (6)

Clearly, $(2, 7)$ is a solution of $(6)$. For each solution $(x, y)$ of $(6)$, the map sends $(x, y)$ to
$(3x + y + 1, 8x + 3y + 4)$, which gives another solution of $(6)$.
(Note: If $(n_i, m_i)$ are all solutions of $(6)$, the sequences $\{n_i\}$ and $\{m_i\}$ are respectively A053141 and A002315 in [7].)

• Case $a = 10$, we consider

$$10n^2 + 10n + 1 = m^2.$$  \hspace{1cm} (7)

Clearly, $(3, 11)$ is a solution of $(7)$. For each solution $(x, y)$ of $(7)$, the map sends $(x, y)$ to
$(19x + 6y + 9, 60x + 19y + 30)$, which gives another solution of $(7)$.
(Note: If $(n_i, m_i)$ are all solutions of $(7)$, the sequence $\{n_i\}$ is A222390 in [7].)

• Case $a = 11$, we consider

$$11n^2 + 11n + 1 = m^2.$$  \hspace{1cm} (8)

Clearly, $(39, 131)$ is a solution of $(8)$. For each solution $(x, y)$ of $(8)$, the map sends $(x, y)$ to
$(199x + 60y + 99, 660x + 199y + 330)$, which gives another solution of $(8)$.
(Note: If $(n_i, m_i)$ are all solutions of $(8)$, the sequences $\{n_i\}$ and $\{m_i\}$ are respectively A105838 and A105837 in [7].)

• Case $a = 12$, we consider

$$12n^2 + 12n + 1 = m^2.$$  \hspace{1cm} (9)

Clearly, $(1, 5)$ is a solution of $(9)$. For each solution $(x, y)$ of $(9)$, the map sends $(x, y)$ to
$(7x + 2y + 3, 24x + 7y + 12)$, which gives another solution of $(9)$.
(Note: If $(n_i, m_i)$ are all solutions of $(9)$, the sequences $\{n_i\}$ and $\{m_i\}$ are respectively A061278 and A001834 in [7].)
Case \( a = 13 \), we consider
\[
13n^2 + 13n + 1 = m^2. \tag{10}
\]

Clearly, \((7, 27)\) is a solution of (10). For each solution \((x, y)\) of (10), the map sends \((x, y)\) to \((649x + 180y + 324, 234x + 649y + 1170)\), which gives another solution of (10).

(Note: If \((n_i, m_i)\) are all solutions of (10), the sequence \(\{n_i\}\) is A104240 in [7].)

Therefore, \(X_n(a)\) is a perfect square for infinitely many \(n\), where \(a\) is not a perfect square. \(\square\)

For \(a > 13\) not a perfect square, the authors consider that \(X_n(a)\) is a perfect square for infinitely many \(n\). We can find the linear map for each solution \((n_0, m_0)\) that gives another solution of (1). But the authors will leave this problem to the interested reader.

From Theorems 1 and 2, we give the following examples for \(a = 1, 3, 5\).

1. \(\prod_{k=1}^{n}(k^4 + k^2 + 1)\) is not a square.
2. \(\prod_{k=1}^{n}(9k^4 - 3k^2 + 1)\) and \(\prod_{k=1}^{n}(25k^4 - 15k^2 + 1)\) are perfect squares for infinitely many \(n\).

Next, we give analogue of \(\Omega_2(n)\) for the product \((2^2 - 1)(3^2 - 1)\ldots(n^2 - 1)\).

**Theorem 3.** The product \(\prod_{k=2}^{n}(k^2 - 1)\) is a perfect square if and only if the triangular number \(T_n\) is a perfect square for \(n > 1\).

**Proof.** The triangular number \(T_n\) is a number obtained by adding all positive integers less than or equal to a given positive integer \(n\), i.e., \(T_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\). We have
\[
\prod_{k=2}^{n}(k^2 - 1) = \prod_{k=2}^{n}(k-1)(k+1)
\]
\[
= \left(\prod_{k=3}^{n-1} k\right)^2 2n(n+1)
\]
\[
= \left(\prod_{k=3}^{n-1} k\right)^2 4T_n.
\]

Thus, this product is a square if and only if \(T_n\) is a square. \(\square\)

The triangular number \(T_n\) is a square (see A001108 in [7]) when the value of \(n\) is 1, 8, 49, 288, 1681, 9800, 57121, 332928, 1940449, 11309768, . . .

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References


[4] Fang, J. H. (2009) Neither \(\prod_{k=1}^n (4k^2 + 1)\) nor \(\prod_{k=1}^n (2k(k - 1) + 1)\) is a perfect square, *Integers*, 9, 177–180.


