A symmetric Diophantine equation involving biquadrates

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Abstract: This paper is concerned with the Diophantine equation $\sum_{i=1}^{n} a_i x_i^4 = \sum_{i=1}^{n} a_i y_i^4$ where $n \geq 3$ and a_i, $i = 1, 2, \ldots, n$, are arbitrary nonzero integers. While a method of obtaining numerical solutions of such an equation has recently been given, it seems that an explicit parametric solution of this Diophantine equation has not yet been published. We obtain a multi-parameter solution of this equation for arbitrary values of a_i and for any positive integer $n \geq 3$, and deduce specific solutions when $n = 3$ and $n = 4$. The numerical solutions thus obtained are much smaller than the integer solutions of such equations obtained earlier.
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This paper is concerned with the Diophantine equation

$$\sum_{i=1}^{n} a_i x_i^4 = \sum_{i=1}^{n} a_i y_i^4, \tag{1}$$

where n is a positive integer ≥ 3 and a_i, $i = 1, 2, \ldots, n$, are arbitrary nonzero integers. We will be particularly interested in the following two special cases of Eq. (1):

$$a_1 x_1^4 + a_2 x_2^4 + a_3 x_3^4 = a_1 y_1^4 + a_2 y_2^4 + a_3 y_3^4, \tag{2}$$

$$\text{and} \quad a_1 x_1^4 + a_2 x_2^4 + a_3 x_3^4 + a_4 x_4^4 = a_1 y_1^4 + a_2 y_2^4 + a_3 y_3^4 + a_4 y_4^4. \tag{3}$$

Izadi and Baghalaghdam [4] have recently given a method of obtaining numerical solutions of the above three equations by first relating their integer solutions to rational points of an elliptic

curve. If the elliptic curve is of positive rank, infinitely many integer solutions of these equations can be obtained. It is, however, to be noted that for each of the three equations (1), (2) and (3), if the elliptic curve concerned is of rank 0, the method given in [4] does not yield infinitely many integer solutions of the equation under consideration.

It seems that for arbitrary integer values of a_i, explicit parametric solutions of Eq. (1), Eq. (2) and Eq. (3) have not been published till now. However, in the special case when the integers a_i are all 1, both Eq. (2) and Eq. (3) have been considered individually as well as part of a larger Diophantine system, and several parametric solutions of these Diophantine equations are to be found scattered through the existing literature (see, for instance, [1, pp. 305–306], [2, pp. 781-782], [3, pp. 653–657]).

We note that all the three equations (1), (2) and (3) are homogeneous equations, and hence any solution in rational numbers of these equations yields, on appropriate scaling, a primitive solution in integers. Therefore it suffices to find rational solutions of these equations.

In this paper we obtain a multi-parameter solution of Eq. (1) for any positive integer $n \geq 3$ and for arbitrary nonzero integer values of a_i, $i = 1, 2, \ldots, n$. Parametric solutions of Eq. (2) and of Eq. (3) may be obtained by taking $n = 3$ and $n = 4$ respectively in the solution of Eq. (1). The numerical solutions of Eq. (2) and of Eq. (3) obtained from these parametric solutions are much smaller than the solutions of these equations obtained in [4] by using elliptic curves.

We note that the aforementioned multi-parameter solution of Eq.(1) yields only trivial solutions of Eq. (1) when $n = 2$ and the question of finding integer solutions of the quartic Diophantine equation

$$a_1 x_1^4 + a_2 x_2^4 = a_1 y_1^4 + a_2 y_2^4, \tag{4}$$

for arbitrary integer values of a_1 and a_2 remains an open problem.

We will now obtain a solution of Eq. (1) by following a general method described in [1] for solving symmetric Diophantine systems. We have made minor modifications in the method so that the solution can be expressed concisely.

We note that if there is any solution of (1) in which $x_i = \pm y_i$ for some i, we can cancel out the terms $a_i x_i^4$ and $a_i y_i^4$ on either side of (1) and, in effect, we get a solution of an equation of type (1) with just $n - 1$ biquadrates on either side. We will therefore obtain only such solutions of Eq. (1) in which $x_i \neq \pm y_i$, $i = 1, 2, \ldots, n$.

To solve Eq. (1), we write,

$$x_i = (f_i + g_i)u + r_i v, \quad y_i = (f_i - g_i)u + r_i v, \ i = 1, 2, \ldots, n, \tag{5}$$

where f_i, g_i, r_i, u and v are arbitrary parameters such that $r_i \neq 0$. Since $x_i - y_i = 2g_i u$, we impose the conditions $u \neq 0$, $g_i \neq 0$, $i = 1, 2, \ldots, n$, so that the solutions that we obtain satisfy the conditions $x_i \neq \pm y_i$, $i = 1, 2, \ldots, n$.

On substituting the values of x_i, y_i given by (5) in Eq. (1) and transposing all terms to the left-hand side, we get

$$\left(\sum_{i=1}^{n} a_i f_i g_i (f_i^2 + g_i^2)\right) u^3 + \left(\sum_{i=1}^{n} a_i g_i r_i (3f_i^2 + g_i^2)\right) u^2 v$$

$$+ 3\left(\sum_{i=1}^{n} a_i f_i g_i r_i^2\right) uv^2 + \left(\sum_{i=1}^{n} a_i g_i r_i^3\right) v^3 = 0. \tag{6}$$

We will now choose the parameters f_i, g_i such that the coefficients of uv^2 and v^3 in Eq. (6) become 0. Equating to 0 the coefficient of v^3 in (6), we get a linear equation in g_i, $i = 1, 2, \ldots, n$, whose complete solution in rational numbers may be written as

$$g_i = (p_i - p_{i+1})/(a_i r_i^3), \ i = 1, 2, \ldots, n, \tag{7}$$

where p_i, $i = 1, 2, \ldots, n$, are arbitrary rational parameters such that $p_i \neq p_{i+1}$ for $i = 1, 2, \ldots, n$ and $p_{n+1} = p_1$. We note that there can only be $n - 1$ independent linear parameters in the solution of the linear equation obtained by equating to 0 the coefficient of v^3 in (6), and hence one of the parameters p_i, $i = 1, 2, \ldots, n$ is actually superfluous but we prefer to write the solution in the symmetric manner given by (7). The condition $p_i \neq p_{i+1}$ for each i ensures that $g_i \neq 0$, $i = 1, 2, \ldots, n$.

With the values of g_i now given by (7), on equating to 0 the coefficient of uv^2 in Eq. (6), we get a linear equation in f_i, $i = 1, 2, \ldots, n$, whose complete solution in rational numbers may be written as

$$f_i = (q_i - q_{i+1})/(a_i g_i r_i^2), \ i = 1, 2, \ldots, n, \tag{8}$$

where q_i, $i = 1, 2, \ldots, n$ are arbitrary rational parameters and $q_{n+1} = q_1$. Here again, one of the parameters q_i, $i = 1, 2, \ldots, n$ is superfluous.

With the values of f_i, g_i, $i = 1, 2, \ldots, n$, being given by (7) and (8), Eq. (6) reduces, on removing the nonzero factor u^2, to a linear equation in u and v whose solution is given by

$$u = \sum_{i=1}^{n} a_i g_i r_i (3f_i^2 + g_i^2), \quad v = -\sum_{i=1}^{n} a_i f_i g_i (f_i^2 + g_i^2). \tag{9}$$

Thus a rational solution of Eq. (1) for arbitrary nonzero integer values of a_i is given by (5) where r_i, $i = 1, 2, \ldots, n$, are arbitrary nonzero parameters, the values of g_i, $i = 1, 2, \ldots, n$, are given by (7) in terms of arbitrary rational parameters p_i, $i = 1, 2, \ldots, n$, such that $p_i \neq p_{i+1}$ for $i = 1, 2, \ldots, n$ and $p_{n+1} = p_1$, the values of f_i, $i = 1, 2, \ldots, n$, are given by (8) in terms of arbitrary rational parameters q_i, $i = 1, 2, \ldots, n$ and $q_{n+1} = q_1$, and the values of u, v are given by (9).

A parametric solution of Eq. (2), obtained by taking $n = 3$, $p_3 = 0$, $q_3 = 0$ and making suitable simplifications in the solution of Eq. (1), is given by

$$x_i = (f_i + g_i)u + r_i v, \ y_i = (f_i - g_i)u + r_i v, \ i = 1, 2, 3, \tag{10}$$

with the values of f_i, g_i, $i = 1, 2, 3$ and u, v being given by

$$f_1 = \lambda p_1 p_2 r_1 (q_1 + q_2), \qquad g_1 = -\mu a_2 a_3 r_2^3 r_3^3 (p_1 + p_2),$$
$$f_2 = \lambda p_1 q_2 r_2 (p_1 + p_2), \qquad g_2 = \mu a_1 a_3 p_2 r_1^3 r_3^3,$$
$$f_3 = \lambda p_2 q_1 r_3 (p_1 + p_2), \qquad g_3 = \mu a_1 a_2 p_1 r_1^3 r_2^3, \tag{11}$$
$$u = \sum_{i=1}^{3} a_i g_i r_i (3f_i^2 + g_i^2), \quad v = -\sum_{i=1}^{3} a_i f_i g_i (f_i^2 + g_i^2),$$

where p_1, p_2, q_1, q_2, r_1, r_2, r_3 are arbitrary nonzero integer parameters such that $p_1 \neq -p_2$ while λ and μ are arbitrary nonzero rational parameters.

As a numerical example, a solution of the equation,

$$x_1^4 + x_2^4 + 61x_3^4 = y_1^4 + y_2^4 + 61y_3^4, \tag{12}$$

obtained by taking $p_1 = 61$, $p_2 = -56$, $q_1 = 61$, $q_2 = -21$, $r_1 = 1$, $r_2 = 2$, $r_3 = -1$, $\lambda = 1/4270$, $\mu = 1/488$ is given by

$$6707^4 + 12802^4 + 61.(3237)^4 = 11227^4 + 6474^4 + 61.(4141)^4.$$

This solution of Eq. (12) consists of relatively small integers as compared to the smallest solution of Eq. (12) given in [4] which involves large integers having 24 digits.

As in the case of Eq. (2), we get a parametric solution of Eq. (3) by taking $n = 4$ in the solution of Eq. (1) given by (5), (7), (8) and (9). As a numerical example, a solution of the equation,

$$x_1^4 + x_2^4 + x_3^4 + 19x_4^4 = y_1^4 + y_2^4 + y_3^4 + 19y_4^4, \tag{13}$$

obtained by taking $n = 4$, $p_1 = 10$, $p_2 = 4$, $p_3 = -4$, $p_4 = -9$, $q_1 = 1$, $q_2 = -2$, $q_3 = 6$, $q_4 = 1$, $r_1 = 1$, $r_2 = 2$, $r_3 = 1$, $r_4 = -1$ in the parametric solution of Eq. (1), is given by

$$576^4 + 220^4 + 527^4 + 19.(159)^4 = 600^4 + 416^4 + 453^4 + 19.(37)^4.$$

This solution is also much smaller than the smallest solution of Eq. (13), consisting of integers having 22 digits, given in [4].

Acknowledgment

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