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Even dimensional rhotrix

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Abstract: A rhotrix is a rhomboidal array of numbers. In many respects, rhotrices are similar to matrices, and matrices, though, are of both even and odd dimensions but only rhotrices of odd dimension are well-known in literature. Even dimensional rhotrix has not been discussed. Therefore, this article introduces rhotrices with even dimension. These rhotrices are a special type of rhotrix where the heart has been extracted. Analysis, examples and some properties of these even-dimensional (heartless) rhotrices are presented and established as algebraic structures, mathematically tractable, and as a contribution to the concept of rhotrix algebra.
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1 Introduction

The possibility of obtaining array of numbers that are in some way, between two-dimensional vectors and (2×2)-dimensional matrices was first discussed by Atanassov and Shannon [4] in their paper: matrix -tertions and noitrets. As an extension of this idea, Ajibade [2] in 2003 introduced objects which are in some ways, between (2×2)-dimensional and (3×3) dimensional matrices. He called such an object a rhotrix. Thus, he defined a rhotrix of dimension three as:

$$R = \{\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \ \in \mathfrak{R}\},$$

where c is known as the heart of the rhotrix R often indicated as $h(R)$. It has been noted that these heart-oriented rhotrices are always of odd dimension. Thus, a rhotrix with dimension n has $|R_n|$ entries where $|R_n| = \frac{1}{2}(n^2+1)$ and $n \in 2\mathbb{Z}^+ + 1$ [2, 12].

The algebra and analysis of rhotrices were first presented in [2]. Thus, addition and multiplication of two heart-based rhotrices were defined as:

$$R + Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{array} \right\rangle$$

and

$$R \circ Q = \left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{array} \right\rangle,$$

respectively. A generalization of this hearty multiplication is given in Mohammed [10] and in Ezegwu *et al* [6].

A row-column multiplication of heart-based rhotrices was proposed by Sani [13] as:

$$R \circ Q = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{array} \right\rangle.$$

A generalization of this row-column multiplication was also later given by Sani [14] as:

$$R_n \circ Q_n = \langle a_{i_1j_1}, c_{i_2j_2} \rangle \circ \langle b_{i_2j_2}, d_{l_2k_2} \rangle = \left\langle \sum_{i_2j_1=1}^{t} (a_{i_1j_1}b_{i_2j_2}), \sum_{l_2k_1=1}^{t-1} (c_{l_1k_1}d_{l_2k_2}) \right\rangle, t = (n+1)/2,$$

where R_n and Q_n are n-dimensional rhotrices (with n rows and n columns). Commenting on this method of multiplication by Sani, Mohammed in [12] says: "a unique expression for the 'rhotrix-heart' cannot be deduced and therefore, this method of rhotrix expression is unsuitable for presenting heart-based rhotrices". This challenge will be addressed in this article.

However, heart-oriented rhotrices are well-known in literature. Mohammed [9] presents additional classifications of heart-based rhotrices as abstract structures of rings, field, integral domain and unique factorization domain. Ezegwu *et al* present a generalized expression for representing n-dimensional heart-oriented rhotrices in a computational environment [6]. Later, Mohammed et al examines the necessary and sufficient condition under which a linear map can be represented over heart-oriented rhotrix [11]. In other papers, Mohammed [10], and Isere [7] present new techniques for expressing rhotrices in a generalized form. Interestingly, Tudunkaya and Manjuola provide a method of constructing finite fields through the use of rhotrices [16]. Significant effort has also been made towards application of heart-oriented rhotrices. Usaini and Mohammed in their article present some properties of rhotrix eigenvalues and eigenvectors [17]. The foregoing are some of the works that have been done to establish these relatively new structures as algebraic structures.

A more flexible nature of array of numbers was seen in the introduction of paraletrix by Aminu and Michael [3]. This structure allows paraletrix to be of different number of rows and columns, thus generalizes rhotrices, which are always of equal number of rows and columns. It is worthy to mention that not every paraletrix has a 'heart' [3]. This subjects therefore, that there can be rhotrices with no heart, and by definition, rhotrices with even dimension [8]. While rhotrices of odd dimension are discussed extensively, those of even dimension are yet to be discussed. The foregoing, therefore, is a motivation for this paper. It should be recalled quickly, that the name rhotrix is as a result of the rhomboid nature of the arrangement of its entries. With or without an entry at the centre, the rhomboid nature is still retained. Therefore, we present rhotrices with even dimension which are rhotrices with a 'missing heart' or 'extracted heart' or better still, 'heartless rhotrices'. These names shall be used interchangeably in this paper.

2 Even dimensional rhotrices

Definition 2.1. *A rhotrix*

$$A = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle : a, b, d, e \in \mathbb{R} \right\}$$

is called a real rhotrix of dimension two.

This is a set of all two even-dimensional (heartless) rhotrices. This even-dimensional rhotrix is a novelty. In this work, we shall simply refer to even-dimensional rhotrices as heartless rhotrices with an acronym hl-rhotrices. Extension in the size of this rhotrix is very possible as examples of some higher hl-rhotrices will be presented in subsequent section. Accordingly, the cardinality of n-dimensional real hl-rhotrix is denoted as $|\hat{R_n}(\mathfrak{R})| = \frac{1}{2}(n^2 + 2n)$, where $n \in 2\mathbb{N}$. This implies that all hl-rhotrices are of even dimension. Therefore, all hl-rhotrices are rhotrices but all rhotrices are not hl-rhotrices.

Remark 2.1. *Recall that the set $\mathbb{N} = \{0, 1, 2, 3, ...\}$. see [1] [5]-The set of natural number or better still the set of non-negative integers. This set is larger than the set of positive integers($\mathbb{Z}^+$).*

3 Operations on hl-rhotrices

In this section, hl-rhotrices of dimension two shall be used for illustrative purpose. However, extension into higher hl-rhotrices is very possible. Some examples of these will be highlighted in the next section.

3.1 Addition of hl-rhotrices

Consider two hl-rhotrices:

$$A = \left\langle \begin{array}{ccc} & a_{11} & \\ a_{21} & & a_{12} \\ & a_{22} & \end{array} \right\rangle ; \; B = \left\langle \begin{array}{ccc} & b_{11} & \\ b_{21} & & b_{12} \\ & b_{22} & \end{array} \right\rangle .$$

We define addition $(+)$ by:

$$A+B=\left\langle\begin{array}{ccc} & a_{11} & \\ a_{21} & & a_{12} \\ & a_{22} & \end{array}\right\rangle+\left\langle\begin{array}{ccc} & b_{11} & \\ b_{21} & & b_{12} \\ & b_{22} & \end{array}\right\rangle=\left\langle\begin{array}{ccc} & a_{11}+b_{11} & \\ a_{21}+b_{21} & & a_{12}+b_{12} \\ & a_{22}+b_{22} & \end{array}\right\rangle$$

It can easily be shown that $(A,+)$ is a commutative group with the zero and negative elements being:

$$0=\left\langle\begin{array}{ccc} & 0 & \\ 0 & & 0 \\ & 0 & \end{array}\right\rangle$$

and

$$-A=\left\langle\begin{array}{ccc} & -a_{11} & \\ -a_{21} & & a_{12} \\ & -a_{22} & \end{array}\right\rangle,$$

respectively.

3.2 Scalar multiplication

The scalar multiplication is defined as follows: If $\lambda \in \Re$ is a scalar and A is a hl-rhotrix, then

$$\lambda A=\lambda\left\langle\begin{array}{ccc} & a_{11} & \\ a_{21} & & a_{12} \\ & a_{22} & \end{array}\right\rangle=\left\langle\begin{array}{ccc} & \lambda a_{11} & \\ \lambda a_{21} & & \lambda a_{12} \\ & \lambda a_{22} & \end{array}\right\rangle.$$

3.3 Multiplication of hl-rhotrices

The multiplicative operation of hl-rhotrices will be the row-column by Sani [13, 14] without incorporating the heart. This method as proposed by Sani is naturally suitable for hl-rhotrices. Hence, we define multiplication as follows:

$$A\circ B=\left\langle\begin{array}{ccc} & a_{11} & \\ a_{21} & & a_{12} \\ & a_{22} & \end{array}\right\rangle\circ\left\langle\begin{array}{ccc} & b_{11} & \\ b_{21} & & b_{12} \\ & b_{22} & \end{array}\right\rangle=\left\langle\begin{array}{ccc} & a_{11}b_{11}+a_{12}b_{21} & \\ a_{21}b_{11}+a_{22}b_{21} & & a_{11}b_{12}+a_{12}b_{22} \\ & a_{21}b_{12}+a_{22}b_{22} & \end{array}\right\rangle.$$

In other words, the multiplication of higher hl-rhotrices is still according to Sani [13, 14] with the empty heart treated as null element or zero-valued element. Treating our rhotrices this way allows us to see the higher dimensional hl-rhotrices as coupled matrices [15] with the lower dimensional squared matrix (minor matrix) coupled in the higher dimensional squared matrix (major matrix).

Remark 3.1. *This modified Sani multiplication removes the unsuitability or non-uniqueness of the heart expression alluded to in [12]. The multiplication of higher hl-rhotrices is by the row-column method except when the row and column with the missing heart coincide.*

It can easily be verified that the set of all hl-rhotrices with multiplicative operation defined this way is a non-commutative algebra.

3.4 Identity element

Consider an hl-rhotrix A of n-dimensional, if I is also an hl-rhotrix of n-dimensional such that:

$$A \circ I = A = I \circ A.$$

Then I is an identity element.

For a 2-dimensional hl-rhotrix, the identity element is given as:

$$I = \left\langle \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array} \right\rangle .$$

Remark 3.2. *A 2-dimensional identity hl-rhotrix is presented for illustration purpose. For higher hl-rhotrices, the identity element is deduced accordingly by allowing entries at the major diagonal to be unity except at the centre which is empty while other entries are zeros. We speak of a major diagonal because we are seeing our hl-rhotrices as coupled matrices.*

3.5 Inverse of an hl-rhotrix

The concept of identity element makes the inverse of an hl-rhotrix meaningful. If for any hl-rhotrix A, we can find another hl-rhotrix X such that

$$A \circ X = X \circ A = I,$$

then X will be the inverse of A.

For example, let

$$A = \left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle$$

then,

$$X = \frac{1}{ae - bd} \left\langle \begin{array}{ccc} & e & \\ -b & & -d \\ & a & \end{array} \right\rangle$$

implies that

$$A^{-1} = \frac{1}{ae - bd} \left\langle \begin{array}{ccc} & e & \\ -b & & -d \\ & a & \end{array} \right\rangle$$

provided $ae \neq bd$. Otherwise, A will be called a singular hl-rhotrix.

Remark 3.3. *It is to be noted that the majority of the heart-based rhotrices with a non-zero heart are non-singular or invertible rhotrices. Not all hl-rhotrices are invertible.*

4 Some examples of higher hl-rhotrices

This section presents some examples of higher dimensional hl-rhotrices. We will be presenting their corresponding coupled matrices along. If R_n is an hl-rhotrix, then R_n^c will denote the corresponding coupled matrix.

(i) A hl-rhotrix of dimension four (R_4) is given by:

$$R_4 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle .$$

Then its corresponding coupled matrix will be presented below:

(ii)

$$R_4^c = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} \\ & c_{11} & & c_{12} & \\ a_{21} & & & & a_{23} \\ & c_{21} & & c_{22} & \\ a_{31} & & a_{32} & & a_{33} \end{bmatrix} .$$

Remark 4.1. *Because of the absence of a heart, we have a surrogate 3×3 matrix (major matrix) coupled with a 2×2 matrix (minor matrix). Representation of heart-based rhotrices(h-rhotrices) into coupled matrices was introduced by Sani [15]. In this case of R_4, the missing heart is in the surrogate matrix.*

(iii) A hl-rhotrix of dimension six (R_6) is given by:

$$R_6 = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ a_{41} & c_{31} & a_{32} & & a_{23} & c_{13} & a_{14} \\ & a_{42} & c_{32} & a_{33} & c_{23} & a_{24} & \\ & & a_{43} & c_{33} & a_{34} & & \\ & & & a_{44} & & & \end{array} \right\rangle .$$

Then its corresponding coupled matrix is:

(iv)

$$R_6^c = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} & & a_{14} \\ & c_{11} & & c_{12} & & c_{13} & \\ a_{21} & & a_{22} & & a_{23} & & a_{24} \\ & c_{21} & & & & c_{23} & \\ a_{31} & & a_{32} & & a_{33} & & a_{34} \\ & c_{31} & & c_{32} & & c_{33} & \\ a_{41} & & a_{42} & & a_{43} & & a_{44} \end{bmatrix} .$$

Remark 4.2. *In this case of R_6, we have a surrogate 3×3 minor matrix and the 4×4 major matrix. The missing heart is in the surrogate 3×3 matrix.*

5 Properties of hl-rhotrices

Lemma 5.1. *Given any m-dimensional hl-rhotrix, $m \in 2\mathbb{N}$ and $m \geq 4$. The following hold:*

(i) The heart will be missing from the major matrix if the dimension (D_m) lies in the set below:

$$\{D_{4+4n} : n \in \mathbb{N}\}.$$

(ii) The major matrix is called a surrogate major matrix if the dimension D_m of a hl-rhotrix lies in the set below:

$$\{D_{4+4n} : n \in \mathbb{N}\}.$$

(iii) The heart will be missing from the minor matrix if the dimension (D_m) lies in the set below:

$$\{D_{6+4n} : n \in \mathbb{N}\}.$$

(iv) The minor matrix is called a surrogate minor matrix if the dimension D_m of a hl-rhotrix lies in the set below:

$$\{D_{6+4n} : n \in \mathbb{N}\}.$$

Proof: The proof follows from section 4. □

5.1 The comparison between heart-based real rhotrices (h-rhotrices) and real hl-rhotrices

The similarities and differences between h-rhotrices and hl-rhotrices are presented below:

	h-rhotrix	hl-rhotrix
1	Equal row and column	Equal row and column
2	The heart exists	The heart does not exist
3	Mostly invertible, provided the heart is not zero	Not all are invertible
4	It gives two squared-coupled matrix	It gives a squared-surrogate coupled matrix
5	Odd dimensional	Even dimensional
6	$\|R_n\| = \frac{1}{2}(n^2 + 1), n \in 2\mathbb{Z}^+ + 1$	$\|R_n\| = \frac{1}{2}(n^2 + 2n), n \in 2\mathbb{N}$
7	$\|R_n\| = m$ where n, m are both odd	$\|R_{n-1}\| = m - 1$ (both even)

Table 1. h-Rhotrices and hl-Rhotrices

5.2 Linear properties of hl-rhotrices

The set A of all hl-rhotrices form a vector space which is spanned by the following vector (rhotrices)

$$I = \left\langle \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 0 & \end{array} \right\rangle; J = \left\langle \begin{array}{ccc} & 0 & \\ 0 & & 1 \\ & 0 & \end{array} \right\rangle; K = \left\langle \begin{array}{ccc} & 0 & \\ 1 & & 0 \\ & 0 & \end{array} \right\rangle; L = \left\langle \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ & 1 & \end{array} \right\rangle.$$

These vectors are linearly independent, therefore, the set $S = \{I, J, K, L\}$ forms a basis for a 2-dimensional hl-rhotrix (R_2).

Any 2-dimensional hl-rhotrix (A),

$$A = \left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle$$

can be written as a linear combination of the rhotrices in S so that

$$A = aI + bK + dK + eL.$$

6 Summary

This article introduced hl-rhotrices which are even-dimensional and established that these are also algebraic structures.

The functions of the heart of a rhotrix are so enormous that it seemed almost impossible to extract it from a rhotrix. This work showed that it is still mathematically tractable to extract the heart of a rhotrix and still obtain an algebraic rhotrix. Moreover, doing so enables one to obtain even dimensional rhotrices. Some properties of these new rhotrices are presented as a contribution to rhotrix algebra. However, it is scholastic to examine further properties of the hl-rhotrices vis a vis the h-rhotrices. The representation of the hl-rhotrices over a linear map will be presented in another paper.

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