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On modular happy numbers

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Abstract: In this paper, we investigate the asymptotic behavior of the sequences generated by iterating the process of summing the modular powers of the decimal digits of a number. In particular, we identify all *modular happy numbers*. A number is called modular happy if the sequence obtained by iterating the process of summing the modular powers of the decimal digits of the number ends with 1.
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1 Motivation

Let $\mathbb{Z}_+$ denote the set of positive integers and let us consider the recursively defined function $\psi : \mathbb{Z}_+ \to \mathbb{Z}_+$ given by

$$\psi(x) = \begin{cases} x & \text{if} \quad x \text{ is single-digit} \\ \psi(\text{sum of the digits of } x) & \text{if} \quad x \text{ is multi-digit} \end{cases}. \tag{1.1}$$

Thus, for example,

$$\psi(8) = 8, \quad \psi(13) = 4, \quad \psi(56) = \psi(11) = 2, \text{and} \quad \psi(271) = \psi(10) = \psi(1) = 1.$$

This function was introduced by the second author who is a pre-med student at the King Saud bin Abdulaziz University for Health Sciences. He used the symbol F. However, we recently found out that such function was investigated by Atanassov [1, 2] (see also the papers cited therein) and denoted by ψ. As such, from here and onward, we follow Atanossov's notation. Furthermore, Bayyati explored sequences $\{\psi_m(n) = \psi(n^m)\}_{n=1}^{\infty}$ for $m = 1, 2, 3, \ldots$ (see the EXCEL-generated Table 1 below) and formulated some observations:

- If n is a multiple of 3, then for $m \geq 2$, $\psi_m(n) = 9$.
- For $m = 1, 2, 3, \ldots, \psi_m(n)$ repeats after 9. Furthermore, the repetition starts earlier whenever m is a multiple of 3.
- For $m \geq 2$, $\psi_{m+6}(n) = \psi_m(n)$ for all n.

x	F	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{25}
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	8	7	5	1	2	4	8	7	5	1	2
3	3	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**
4	4	7	1	4	7	1	4	7	1	4	7	1	4
5	5	7	8	4	2	1	5	7	8	4	2	1	5
6	6	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**
7	7	4	1	7	4	1	7	4	1	7	4	1	7
8	8	1	8	1	8	1	8	1	8	1	8	1	8
9	9	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**
10	1	1	1	1	1	1	1	1	1	1	1	1	1
11	2	4	8	7	5	1	2	4	8	7	5	1	2
12	3	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**

Table 1. $F_m(x) = \psi_m(x)$ for $m = 1, \ldots, 25$ and $x = 1, \ldots, 12$

In this paper, we go a step further and iterate the following process.

$$f(n) = \sum_{j=0}^{k} \psi_m(d_j) = \sum_{j=0}^{k} \psi\left(d_j^m\right) \quad \text{where} \quad n = d_k \cdots d_1 d_0, \quad m = 1, 2, 3, \ldots \tag{1.2}$$

We prove, in Section 3, that the sequences generated in this way converge, more precisely are eventually constant. Following the spirit of happy numbers [3, p. 374], if the limit is 1, we call the number n *modular happy*. Moreover, preliminary results are established in Section 2, and further applications of the proof ideas will be introduced in Section 4. We conclude in Section 5 with a summary and future directions.

2 Preliminary results

To be on the same page, we recall the following standard definitions; see, for example, Rosen [4, p. 241] and Weisstein [5].

Definition 2.1. *Let q be a positive integer. A sequence a_n is said to be* ***periodic of period*** *q if*

$$a_{n+q} = a_n \quad \textit{for } n = 1, 2, 3, \ldots$$

If q is the smallest such integer, it is called ***minimal period****.*

Definition 2.2. *Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \bmod m = b \bmod m$, where $a \bmod m = a - m\lfloor a/m \rfloor$ where $\lfloor \cdot \rfloor$ denotes the floor function, i.e., the greatest integer function.*

For the proof of our main result in this section, we shall need the following well-known lemma; see Rosen (2012, p. 242).

Lemma 2.1. *Let m be a positive integer and let a and b be integers. Then*

$$(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$$

and

$$ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m.$$

Next, we state and prove the observations mentioned in Section 1. However, we first establish the following equivalency.

Lemma 2.2. *For any positive integer n,*

$$\psi(n) \equiv n \bmod 9.$$

More precisely,

$$\psi(n) = \begin{cases} n \bmod 9 & \textit{if} \quad n \bmod 9 \neq 0 \\ 9 & \textit{if} \quad n \bmod 9 = 0 \end{cases}.$$

Proof. If n is single-digit, then the equivalency is straight forward. So, assume $n = d_k \cdots d_2 d_0$. By the expanded notation,

$$n = \sum_{j=0}^{k} d_j 10^j = \sum_{j=1}^{k} d_j \left(10^j - 1\right) + \sum_{j=0}^{k} d_j.$$

Since the first sum is a multiple of 9,

$$n \bmod 9 = \left(\sum_{j=0}^{k} d_j \right) \bmod 9.$$

Hence, the result follows. □

Remark 2.1. *The above equivalency lemma was also observed by Atanassov [2, p. 9]*

Theorem 2.1. *Let $m,\ n$ be positive integers and recall $\{\psi_m(n) = \psi\left(n^m\right)\}_{n=1}^{\infty}$ for $m = 1, 2, \ldots$. Then*

1. *If n is a multiple of 3, then for $m \geq 2$, $\psi_m(n) = 9$.*

2. *For $m = 1, 2, 3, \ldots$, the sequence $\{\psi_m(n)\}_{n=1}^{\infty}$ is periodic of period 9.*

3. *If m is a multiple of 3, then the minimal period is 3.*

4. *For $m \geq 2$, $\psi_{m+6}(n) = \psi_m(n)$ for all n.*

5. *For $m \geq 2$, $\psi_m(n) \neq 6$ for all $n \geq 1$.*

Proof. As seen below, Lemma 2.2 plays a key role in our proof.

1. Suppose $n = 3\ell$ for some positive integer ℓ. Since $(3\ell)^k$ is a multiple of 9 for $k \geq 2$,
$$(3\ell)^k \bmod 9 = 0,$$
and so $\psi_k(n) = 9$.

2. With the understanding that, $\psi(n) = \psi_1(n)$, for any $k \geq 1$, by Lemma (2.1),
$$(n+9)^k \bmod 9 = (n + 9 \bmod 9)^k \bmod 9 = (n \bmod 9)^k \bmod 9.$$
Therefore, $\psi_k(n+9) = \psi_k(n)$ for all $n \geq 1$.

3. Assume k is a multiple of 3. Then
$$\begin{aligned}(n+3)^k \bmod 9 &= (n+3)^{3\ell} \bmod 9 = \left((n+3)^3\right)^\ell \bmod 9\\ &= \left(n^3 + 9n^2 + 27n + 27\right)^\ell \bmod 9 = \left(n^3\right)^\ell \bmod 9\\ &= n^k \bmod 9.\end{aligned}$$

4. Notice that $\psi_6(n+3) = \psi_6(n)$ and $\{n^6\}_{n=1}^{\infty} \bmod 9 = \{1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots\}$. Consequently,
$$n^{k+6} \bmod 9 = n^6 n^k \bmod 9 = n^k \bmod 9.$$

5. This part follows from Part (4) and the fact that $\psi_k(n) \neq 6$ for $k = 2, 3, ..., 7$.

□

Remark 2.2. *Aside from the terminology, Theorem 2.1 was also observed by Atanassov [2].*

3 Main results

For each positive integer m and in light of Equation (1.2), consider the sequence $\{x_{i+1}\}_0^\infty$ defined by

$$\begin{aligned}x_0 &= d_k \cdots d_1 d_0 = \sum_{j=0}^{k} d_j 10^j\\ x_1 &= f(x_0) = \sum_{j=0}^{k} \psi\left(d_j^m\right)\\ &\vdots \quad \vdots\\ x_{i+1} &= f(x_i), \quad \text{for} \quad i = 0, 1, 2, \ldots\end{aligned} \tag{3.1}$$

With that in mind, our first main result in this section reads as follows:

Theorem 3.1. *For each* $m = 1, 2, 3, \ldots, x_i$ *is eventually less than* 10.

Proof. Observe that if x_0 is a power of 10, then $x_i = 1$ for all $i \geq 1$, and hence all powers of 10 are modular happy. Furthermore, if $x_0 = 9$, then $x_i = 9$ for all $i \geq 1$, and so multiples of 9 are not modular happy.

If $m = 1$, then

$$\begin{aligned} f(n) &= n \quad \text{if} \quad k = 0, \text{ i.e., } n = d_0 \\ f(n) &= \sum_{j=0}^{k} d_j < \sum_{j=0}^{k} d_j 10^j = n \end{aligned}$$

Hence, by the Monotonic Convergence Theorem, the result follows.

To this end, assume that $m \geq 2$ and $n = \sum_{j=0}^{k} d_j 10^j$. Then

$$\begin{aligned} f(n) &= \sum_{j=0}^{k} \psi\left(d_j^m\right) \leq 9(k+1) \\ &\leq n \quad \text{if} \quad n \geq 9(k+1). \end{aligned}$$

In particular, if $k = 1$, then $f(n) < n$ if $n \geq 19$. But, if $n = 1d_0$, then, by Theorem 2.1, $f(n) < n$ if $n \geq 10$.

Moreover, if $k \geq 2$, then

$$\begin{aligned} f\left(d_k d_{k-1} \cdots d_0\right) &= \psi\left(d_k^m\right) + f\left(d_{k-1} \cdots d_0\right) \\ &\leq 9 + d_{k-1} \cdots d_0 \qquad \text{assuming the inquality holds for } k-1 \\ &= 9 + \left(n - d_k 10^k\right) < n \end{aligned}$$

Therefore, by the Principle of Mathematical Induction, $f(n) < n$ for all $k \geq 1$. This completes the proof. □

The next result constitutes our second main result in this section. It characterizes the asymptotic behavior of all solution of Equation (3.1).

Theorem 3.2. *If* $m \equiv 2, 5 \bmod 6$*, then every solution of Equation (3.1) is either eventually constant or eventually periodic of period* 2*. Otherwise, every solution of Equation (3.1) is eventually constant.*

Proof. In light of Theorem 2.1, it is enough to study $m = 1, 2, 3, 4, 5, 6, 7$.

If $m = 1$, then every solution of Equation (3.1) is eventually constant. The possible constants are $1, 2, \ldots, 9$. This is true because $f(n) = n$ for all $n \in [1, 9] \cap \mathbb{Z}_+$.

If $m = 2$, then every solution of Equation (3.1) is eventually constant, namely $1, 9$, or eventually periodic, namely $\{4, 7\}$. This is true because

$$\begin{aligned} &f(1) = 1, \quad f(2) = 4, \quad f(3) = 9, \\ &f(4) = 7, \quad f(5) = 7, \quad f(6) = 9, \\ &f(7) = 4, \quad f(8) = 1, \quad \text{and} \quad f(9) = 9. \end{aligned}$$

If $m = 3$, then every solution of Equation (3.1) is eventually constant, namely $1, 8, 9$. This is true because

$$f(1) = 1, \quad f(2) = 8, \quad f(3) = 9,$$
$$f(4) = 1, \quad f(5) = 8, \quad f(6) = 9,$$
$$f(7) = 1, \quad f(8) = 8, \quad \text{and} \quad f(9) = 9.$$

If $m = 4$, then every solution of Equation (3.1) is eventually constant, namely $1, 4, 7, 9$. This holds true because

$$f(1) = 1, \quad f(2) = 7, \quad f(3) = 9,$$
$$f(4) = 4, \quad f(5) = 4, \quad f(6) = 9,$$
$$f(7) = 7, \quad f(8) = 1, \quad \text{and} \quad f(9) = 9.$$

If $m = 5$, then every solution of Equation (3.1) is eventually constant, namely $1, 8, 9$, or eventually periodic, namely $\{2, 5\}$ or $\{4, 7\}$. This holds true because

$$f(1) = 1, \quad f(2) = 5, \quad f(3) = 9,$$
$$f(4) = 7, \quad f(5) = 2, \quad f(6) = 9,$$
$$f(7) = 4, \quad f(8) = 8, \quad \text{and} \quad f(9) = 9.$$

If $m = 6$, then every solution of Equation (3.1) is eventually constant, namely $1, 9$. This holds true because

$$f(1) = 1, \quad f(2) = 1, \quad f(3) = 9,$$
$$f(4) = 1, \quad f(5) = 1, \quad f(6) = 9,$$
$$f(7) = 1, \quad f(8) = 1, \quad \text{and} \quad f(9) = 9.$$

If $m = 7$, then every solution of Equation (3.1) is eventually constant, namely $1, 2, 4, 5, 7, 8, 9$. This holds true because

$$f(1) = 1, \quad f(2) = 2, \quad f(3) = 9,$$
$$f(4) = 4, \quad f(5) = 5, \quad f(6) = 9,$$
$$f(7) = 7, \quad f(8) = 8, \quad \text{and} \quad f(9) = 9.$$

This completes the proof. □

Remark 3.1. *Clearly, the modular happy numbers are the pre-images of* 1. *For example, if* $m = 2$*, the first modular happy numbers less than or equal to* 100 *are given by*

$$\{1, 8, 10, 13, 14, 15, 16, 19, 22, 27, 31, 38, 41, 44, 45, 48, 51,$$
$$54, 55, 58, 61, 68, 72, 77, 80, 83, 84, 85, 86, 89, 91, 98, 100\}$$

4 Applications

The ideas introduced in the proof of Theorem 2.1 can be utilized to investigate the asymptotic behavior of related sequences. For instance, consider the sequence defined recursively as follows.

$$\begin{aligned} x_0 &= d_k \cdots d_1 d_0 = \sum_{j=0}^{k} d_j 10^j \\ x_1 &= g(x_0) = d_0 + \sum_{j=1}^{k} d_j^j \\ \vdots & \quad \vdots \\ x_{i+1} &= g\left(x_i\right), \quad \text{for} \quad i = 0, 1, 2, \ldots. \end{aligned} \tag{4.1}$$

Observe that

$$\begin{aligned} g(n) &= n \quad \text{if} \quad k = 0, \text{ i.e., } n = d_0 \\ g(n) &= d_0 + \sum_{j=1}^{k} d_j^j = d_0 + \sum_{j=1}^{k} d_j d_j^{j-1} < d_0 + \sum_{j=1}^{k} d_j 10^j = n \quad \text{if} \quad k = 0. \end{aligned}$$

Hence, every solution of Equation (4.1) is eventually constant.

5 Conclusion

To identify modular happy number, we investigated the asymptotic behavior of all solutions of Equation (3.1). Furthermore, the same proving tools were applied to Equation (4.1). By doing so, we opened the door for further studies. Indeed, there is myriad of potential investigations. To exemplify, consider the sequence defined recursively as follows.

$$\begin{aligned} x_0 &= d_k \cdots d_1 d_0 = \sum_{j=0}^{k} d_j 10^j \\ x_1 &= h(x_0) = d_k + \sum_{j=0}^{k-1} d_j^{k-j} \\ \vdots & \quad \vdots \\ x_{i+1} &= h\left(x_i\right), \quad \text{for} \quad i = 0, 1, 2, \ldots. \end{aligned} \tag{5.1}$$

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