The ternary Goldbach problem 
with prime numbers of a mixed type

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Abstract: In the present paper we prove that every sufficiently large odd integer \( N \) can be represented in the form

\[ N = p_1 + p_2 + p_3, \]

where \( p_1, p_2, p_3 \) are primes, such that \( p_1 = x^2 + y^2 + 1 \), \( p_2 = [n^c] \).

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1 Notations

Let \( N \) be a sufficiently large odd integer. The letter \( p \), with or without subscript, will always denote prime numbers. Let \( A > 100 \) be a constant. By \( \varepsilon \) we denote an arbitrary small positive number, not the same in all appearances. The relation \( f(x) \ll g(x) \) means that \( f(x) = O(g(x)) \).

As usual \([t]\) and \(\{t\}\) denote the integer part, respectively, the fractional part of \(t\). Instead of \( m \equiv n \pmod{k} \) we write for simplicity \( m \equiv n \pmod{k} \). As usual \( e(t)=\exp(2\pi it) \). We denote by \((d, q), [d, q]\) the greatest common divisor and the least common multiple of \(d \) and \(q \) respectively.

As usual \( \varphi(d) \) is Euler’s function; \( \mu(d) \) is Möbius’ function; \( r(d) \) is the number of solutions of the equation \( d = m_1^2 + m_2^2 \) in integers \( m_j \); \( \chi(d) \) is the non-principal character modulo 4 and \( L(s, \chi) \) is the corresponding Dirichlet’s \(L\)-function. By \( c_0 \) we denote some positive number, not necessarily the same in different occurrences. Let \( c \) be a real constant such that \( 1 < c < 73/64 \).
Denote
\[\gamma = 1/c;\]  
\[D = \frac{N^{1/2}}{(\log N)^A};\]  
\[\psi(t) = \{t\} - 1/2;\]  
\[\theta_0 = \frac{1}{2} - \frac{1}{4}e \log 2 = 0.0289...;\]  
\[\mathcal{S}_{d,l}(N) = \prod_{p\nmid d} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p\mid d} \left(1 - \frac{1}{(p-1)^2}\right) \times \prod_{p\mid dN} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p\nmid dN} \left(1 + \frac{1}{p-1}\right);\]  
\[\mathcal{S}(N) = \prod_{p\mid N} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p\nmid N} \left(1 + \frac{1}{(p-1)^3}\right);\]  
\[\mathcal{S}_\Gamma(N) = \pi \mathcal{S}(N) \prod_{p\mid (N-1)} \left(1 + \chi(p) \frac{p-3}{p(p^2-3p+3)}\right) \prod_{p\mid N} \left(1 + \chi(p) \frac{1}{p(p-1)}\right) \times \prod_{p\mid (N-1)} \left(1 + \chi(p) \frac{2p-3}{p(p^2-3p+3)}\right);\]  
\[\Delta(t, h) = \max_{y \leq t} \max_{(l, h) = 1} \left| \sum_{\substack{p \leq y \mod (l, h) \equiv 1 \mod l \neq 0}} \log p - \frac{y}{\varphi(h)} \right| .\]

2 Introduction and statement of the result

In 1937, I. M. Vinogradov [15] solved the ternary Goldbach problem. He proved that for a sufficiently large odd integer \(N\)
\[\sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 = \frac{1}{2} \mathcal{S}(N) N^2 + O \left(\frac{N^2}{(\log^A N)}\right),\]
where \(\mathcal{S}(N)\) is defined by (6) and \(A > 0\) is an arbitrarily large constant.

In 1953, Piatetski-Shapiro [9] proved that for any fixed \(c \in (1, 12/11)\) the sequence
\[([n^c])_{n \in \mathbb{N}}\]
contains infinitely many prime numbers. Such prime numbers are named in honor of Piatetski-Shapiro. The interval for \(c\) was subsequently improved many times and the best result up to now belongs to Rivat and Wu [10] for \(c \in (1, 243/205)\).
In 1992, A. Balog and J. P. Friedlander [1] considered the ternary Goldbach problem with variables restricted to Piatetski-Shapiro primes. They proved that, for any fixed \(1 < c < 21/20\), every sufficiently large odd integer \(N\) can be represented in the form

\[ N = p_1 + p_2 + p_3, \]

where \(p_1, p_2, p_3\) are primes, such that \(p_k = [n_c^k]\), \(k=1,2,3\). Rivat [10] extended the range to \(1 < c < 199/188\); Kumchev [7] extended the range to \(1 < c < 53/50\). Jia [5] used a sieve method to enlarge the range to \(1 < c < 16/15\).

Furthermore, Kumchev [7] proved that for any fixed \(1 < c < 73/64\) every sufficiently large odd integer may be written as the sum of two primes and prime number of type \(p = [n_c^c]\).

On the other hand, in 1960, Linnik [8] showed that there exist infinitely many prime numbers of the form \(p = x^2 + y^2 + 1\), where \(x\) and \(y\) are integers. In 2010 Tolev [14] proved that every sufficiently large odd integer \(N\) can be represented in the form

\[ N = p_1 + p_2 + p_3, \]

where \(p_1, p_2, p_3\) are primes, such that \(p_k = x_2^k + y_2^k + 1\), \(k=1,2,3\). In 2017 Teräväinen [12] improved Tolev’s result for primes \(p_1, p_2, p_3\), such that \(p_1 = x_2^2 + y_2^2 + 1\), \(p_2 = [n_c^c]\).

Recently the author [2] proved that there exist infinitely many arithmetic progressions of three different primes \(p_1, p_2, p_3 = 2p_2 - p_1\) such that \(p_1 = x^2 + y^2 + 1\), \(p_3 = [n_c^c]\).

Define

\[ \Gamma(N) = \sum_{p_1+p_2+p_3=N \atop p_2=[n_c^c]} r(p_1-1)p_2^{1-\gamma} \log p_1 \log p_2 \log p_3. \quad (10) \]

Motivated by these results we shall prove the following theorem.

**Theorem 1.** Assume that \(1 < c < 73/64\). Then the asymptotic formula

\[ \Gamma(N) = \frac{\gamma}{2} \mathcal{G}_\Gamma(N) N^2 + O\left(N^2 (\log N)^{-\theta_0} (\log \log N)^6\right), \]

holds. Here \(\gamma, \theta_0\) and \(\mathcal{G}_\Gamma(N)\) are defined by (1), (4) and (7).

Bearing in mind that \(\mathcal{G}_\Gamma(N) \gg 1\) for \(N\) odd, from Theorem 1 it follows that for any fixed \(1 < c < 73/64\) every sufficiently large odd integer \(N\) can be written in the form

\[ N = p_1 + p_2 + p_3, \]

where \(p_1, p_2, p_3\) are primes, such that \(p_1 = x^2 + y^2 + 1\), \(p_2 = [n_c^c]\).

The asymptotic formula obtained for \(\Gamma(N)\) is the product of the individual asymptotic formulas

\[ \sum_{p_1+p_2+p_3=N} r(p_1-1) \log p_1 \log p_2 \log p_3 \sim \frac{1}{2} \mathcal{G}_\Gamma(N) N^2 \]

and

\[ \frac{1}{N} \sum_{p \leq N \atop p=[n_c^c]} p^{1-\gamma} \log p \sim \gamma. \]

The proof of Theorem 1 follows the same ideas as the proof in [2].
3 Outline of the proof

Using (10) and well-known identity \( r(n) = 4 \sum_{d|n} \chi(d) \) we find

\[
\Gamma(N) = 4(\Gamma_1(N) + \Gamma_2(N) + \Gamma_3(N)),
\]

where

\[
\begin{align*}
\Gamma_1(N) &= \sum_{p_1+p_2+p_3=N \atop p_2=[a^c]} \left( \sum_{d|p_1-1 \atop d \leq D} \chi(d) \right) p_1^{1-\gamma} \log p_1 \log p_2 \log p_3, \\
\Gamma_2(N) &= \sum_{p_1+p_2+p_3=N \atop p_2=[a^c]} \left( \sum_{D<d<N/D} \chi(d) \right) p_1^{1-\gamma} \log p_1 \log p_2 \log p_3, \\
\Gamma_3(N) &= \sum_{p_1+p_2+p_3=N \atop p_2=[a^c]} \left( \sum_{d|p_1-1 \atop d \geq N/D} \chi(d) \right) p_1^{1-\gamma} \log p_1 \log p_2 \log p_3.
\end{align*}
\]

In order to estimate \( \Gamma_1(N) \) and \( \Gamma_3(N) \) we have to consider the sum

\[
I_{d,l,J}(N) = \sum_{p_1+p_2+p_3=N \atop p_1 \equiv l (d) \atop p_2=[a^c]} p_1^{1-\gamma} \log p_1 \log p_2 \log p_3,
\]

where \( d \) and \( l \) are coprime natural numbers, and \( J \subset [1,N] \). The left and the right side of the interval \( J \), we shall denote with \( J_1 \) and \( J_2 \), i.e. \( J = (J_1, J_2) \). If \( J = [1,N] \) then we write for simplicity \( I_{d,l}(N) \). We apply the circle method. Clearly

\[
I_{d,l,J}(N) = \int_0^1 S_{d,l,J}(\alpha)S(\alpha)S_c(\alpha)e(-N\alpha)d\alpha,
\]

where

\[
\begin{align*}
S_{d,l,J}(\alpha) &= \sum_{p_1+p_2+p_3=N \atop (p_1,l) = 1} e(\alpha p) \log p, \\
S(\alpha) &= S_{1,1:1,N}(\alpha), \\
S_c(\alpha) &= \sum_{p \leq N \atop (p,q)=1} p^{1-\gamma} e(\alpha p) \log p.
\end{align*}
\]

We define major and minor arcs by

\[
E_1 = \bigcup_{q \leq Q} \bigcup_{a=0 \atop (a,q)=1}^{q-1} \left[ \frac{a}{q} - \frac{1}{q\tau}, \frac{a}{q\tau} + \frac{1}{q\tau} \right], \quad E_2 = \left[ \frac{1}{\tau}, 1 + \frac{1}{\tau} \right] \setminus E_1,
\]

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where
\[ Q = (\log N)^B, \quad \tau = NQ^{-1}, \quad A > 4B + 3, \quad B > 14. \] (21)

Then we have the decomposition
\[ I_{d,l,J}(N) = I^{(1)}_{d,l,J}(N) + I^{(2)}_{d,l,J}(N), \] (22)

where
\[ I^{(i)}_{d,l,J}(N) = \int_{E_i} S_{d,l,J}(\alpha)S(\alpha)S_c(\alpha)e(-N\alpha)d\alpha, \quad i = 1, 2. \] (23)

We shall estimate \( I^{(1)}_{d,l,J}(N), \Gamma_3(N), \Gamma_2(N) \) and \( \Gamma_1(N) \) respectively in the sections 4, 5, 6 and 7. In section 8 we shall complete the proof of the Theorem.

4 Asymptotic formula for \( I^{(1)}_{d,l,J}(N) \)

We have
\[ I^{(1)}_{d,l,J}(N) = \sum_{q \leq Q} \sum_{\alpha = 0}^{\tau} H(a, q), \] (24)

where
\[ H(a, q) = \int_{-1/q\tau}^{1/q\tau} S_{d,l,J}(\alpha + \alpha) S(\alpha) S_c(\alpha) e(-N(\alpha + \alpha)) d\alpha. \] (25)

On the other hand,
\[ S_{d,l,J}(\alpha + \alpha) = \sum_{1 \leq m \leq q} \epsilon\left(\frac{am}{q}\right) T(\alpha) + O(q \log N), \] (26)

where
\[ T(\alpha) = \sum_{\substack{p \leq \ell \mid (d) \mid d, q} \epsilon(\alpha p) \log p. \]

According to the Chinese remainder theorem there exists integer \( f = f(l, m, d, q) \) such that \( (f, [d, q]) = 1 \) and
\[ T(\alpha) = \sum_{\substack{p \leq \ell \mid (f, d, q) \mid (d, q)} \epsilon(\alpha p) \log p. \]
Applying Abel’s transformation we obtain

\[
T(\alpha) = - \int_{J_1}^{J_2} \left( \sum_{p \leq t} \log p \right) \frac{d}{dt} (e(\alpha t)) dt + \left( \sum_{p \in J \setminus \{d, q\}} \log p \right) e(\alpha J_2)
\]

\[
= - \int_{J_1}^{J_2} \left( \frac{t - J_1}{\varphi([d, q])} + \mathcal{O}(\Delta(J_2, [d, q])) \right) \frac{d}{dt} (e(\alpha t)) dt
\]

\[
+ \left( \frac{J_2 - J_1}{\varphi([d, q])} + \mathcal{O}(\Delta(J_2, [d, q])) \right) e(\alpha J_2)
\]

\[
= \frac{1}{\varphi([d, q])} \int_{J_1}^{J_2} e(\alpha t) dt + \mathcal{O}\left( (1 + |\alpha|(J_2 - J_1)) \Delta(J_2, [d, q]) \right).
\]

(27)

We use the well known formula

\[
\int_{J_1}^{J_2} e(\alpha t) dt = M_J(\alpha) + \mathcal{O}(1),
\]

(28)

where

\[
M_J(\alpha) = \sum_{m \in J} e(\alpha m).
\]

Bearing in mind that \( |\alpha| \leq 1/q \tau \) and \( J \subset (1, N) \), from (21), (27) and (28) we get

\[
T(\alpha) = \frac{M_J(\alpha)}{\varphi([d, q])} + \mathcal{O}\left( \left( 1 + \frac{Q}{q} \right) \Delta(N, [d, q]) \right).
\]

(29)

From (26) and (29) it follows

\[
S_{d, l; J} \left( \frac{a}{q} + \alpha \right) = \frac{c_d(a, q, l)}{\varphi([d, q])} M_J(\alpha) + \mathcal{O}(Q(\log N) \Delta(N, [d, q])),
\]

(30)

where

\[
c_d(a, q, l) = \sum_{1 \leq m \leq q \atop m \equiv l (\mod d, q)} e\left( \frac{am}{q} \right).
\]

We shall find asymptotic formula for \( S_c \left( \frac{a}{q} + \alpha \right) \). From (19) we have

\[
S_c(\alpha) = \sum_{p \leq N} p^{1-\gamma} \left( [\gamma] - \psi(-p^\gamma) \right) e(\alpha p) \log p
\]

\[
= \Omega(\alpha) + \Sigma(\alpha),
\]

(31)

where

\[
\Omega(\alpha) = \sum_{p \leq N} p^{1-\gamma} \left( \psi(p + 1) - \psi(p) \right) e(\alpha p) \log p,
\]

(32)

\[
\Sigma(\alpha) = \sum_{p \leq N} p^{1-\gamma} \left( \psi(-p^\gamma) - \psi(p^\gamma) \right) e(\alpha p) \log p.
\]

(33)
According to Kumchev ([7], Theorem 2) for $64/73 < \gamma < 1$ uniformly in $\alpha$ we have that
\[ \Sigma \left( \frac{a}{q} + \alpha \right) \ll N^{1-\epsilon}. \] (34)

On the other hand,
\[ (p + 1)^\gamma - p^\gamma = \gamma p^{\gamma - 1} + O \left( p^{\gamma - 2} \right). \] (35)

The formulas (32) and (35) give us
\[ \Omega(\alpha) = \gamma S(\alpha) + O(N^c), \] (36)

where $S(\alpha)$ is defined by (18).

According to ([6], Lemma 3, §10) we have
\[ S\left( \frac{a}{q} + \alpha \right) = \mu(q) \phi(q) M(J(\alpha)) + O \left( Ne^{-c_0 \sqrt{\log N}} \right), \] (37)

where
\[ M(\alpha) = \sum_{m \leq N} e(\alpha m). \]

Bearing in mind (31), (34), (36) and (37) we obtain
\[ S_c\left( \frac{a}{q} + \alpha \right) = \gamma \frac{\mu(q)}{\phi(q)} M(\alpha) + O \left( Ne^{-c_0 \sqrt{\log N}} \right). \] (38)

Furthermore, we need the trivial estimates
\[ \left| S_{d,l,J} \left( \frac{a}{q} + \alpha \right) \right| \ll \frac{N \log N}{d}, \quad \left| S \left( \frac{a}{q} + \alpha \right) \right| \ll N, \quad |M(\alpha)| \ll N, \quad |\mu(q)| \ll 1. \] (39)

By (30), (37) – (39) and the well-known inequality $\varphi(n) \gg n(\log \log n)^{-1}$ we find
\[ S_{d,l,J} \left( \frac{a}{q} + \alpha \right) S \left( \frac{a}{q} + \alpha \right) S_c \left( \frac{a}{q} + \alpha \right) e\left( -N \left( \frac{a}{q} + \alpha \right) \right) \]
\[ = \gamma \frac{c_d(a,q,l)}{\varphi([d,q]) \varphi(q)} \mu^2(q) M_J(\alpha) M^2(\alpha) e\left( -N \left( \frac{a}{q} + \alpha \right) \right) + O \left( \frac{N^3}{d} e^{-c_0 \sqrt{\log N}} \right) \]
\[ + O \left( \frac{N Q \log N}{q^2} \Delta(N, [d, q]) \right). \] (40)

Having in mind (21), (25) and (40) we get
\[ H(a, q) = \gamma \frac{c_d(a,q,l)}{\phi([d,q]) \phi(q)} \mu^2(q) e\left( -N \frac{a}{q} \right) \int_{-1/q}^{1/q} M_J(\alpha) M^2(\alpha) e(-N\alpha) d\alpha \]
\[ + O \left( \frac{N^2}{qd} e^{-c_0 \sqrt{\log N}} \right) + O \left( \frac{N Q^2 \log^2 N}{q^3} \Delta(N, [d, q]) \right). \] (41)
Taking into account (24), (41) and following the method in [13] we obtain

\[
I_{d,l,J}^{(1)}(N) = \gamma \frac{\mathcal{S}_{d,l}(N)}{\varphi(d)} \sum_{m_1 + m_2 + m_3 = N \atop m_1 \in J} 1 + \mathcal{O} \left( \frac{N^2}{d} (\log N) \sum_{q > Q} \frac{(d, q) \log q}{q^2} \right)
+ \mathcal{O} \left( N^2 (\log N) \sum_{q \leq Q} \frac{\Delta(N, [d, q])}{q^2} \right)
+ \mathcal{O} \left( \frac{N^2}{d} e^{-c_0 \sqrt{\log N}} \right),
\]

(42)

where \( \mathcal{S}_{d,l}(N) \) is defined by (5).

5 Upper bound for \( \Gamma_3(N) \)

Consider the sum \( \Gamma_3(N) \).

Since

\[
\sum_{d \mid p_1 - 1 \atop d \geq N/D} \chi(d) = \sum_{m \leq (p_1 - 1)D/N} \chi\left(\frac{p_1 - 1}{m}\right) = \sum_{j = \pm 1} \chi(j) \sum_{m \leq \left(\frac{p_1 - 1}{D/N}\right) \atop \left(\frac{1}{j}\right) (4)} 1
\]

then from (14) and (15) it follows

\[
\Gamma_3(N) = \sum_{m < D \atop 2 \mid m} \sum_{j = \pm 1} \chi(j) I_{4m,1+jm;J_m}(N),
\]

where \( J_m = [1 + mN/D, N] \).

Therefore from (22) we get

\[
\Gamma_3(N) = \Gamma_3^{(1)}(N) + \Gamma_3^{(2)}(N),
\]

(43)

where

\[
\Gamma_3^{(\nu)}(N) = \sum_{m < D \atop 2 \mid m} \sum_{j = \pm 1} \chi(j) I_{4m,1+jm;J_m}^{(\nu)}(N), \quad \nu = 1, 2.
\]

(44)

Let us consider first \( \Gamma_3^{(2)}(N) \). Bearing in mind (23) for \( i = 2 \) and (44) for \( \nu = 2 \) we have

\[
\Gamma_3^{(2)}(N) = \int_{E_2} K(\alpha) S(\alpha) S_c(\alpha) e(-N\alpha) d\alpha,
\]

where

\[
K(\alpha) = \sum_{m < D \atop 2 \mid m} \sum_{j = \pm 1} \chi(j) S_{4m,1+jm;J_m}(\alpha).
\]

(45)

Using Cauchy’s inequality we obtain

\[
\Gamma_3^{(2)}(N) \leq \sup_{\alpha \in E_2 \setminus \{1\}} |S_c(\alpha)| \int_{E_2} |K(\alpha) S(\alpha)| d\alpha + \mathcal{O}(N^\varepsilon)
\]

\[
\leq \sup_{\alpha \in E_2 \setminus \{1\}} |S_c(\alpha)| \left( \int_0^1 |K(\alpha)|^2 d\alpha \right)^{1/2} \left( \int_0^1 |S(\alpha)|^2 d\alpha \right)^{1/2} + \mathcal{O}(N^\varepsilon).
\]

(46)
From (31) and (36) we have

\[ S_c(\alpha) = \gamma S(\alpha) + \Sigma(\alpha) + O(N^\varepsilon), \quad (47) \]

where \( S(\alpha) \) and \( \Sigma(\alpha) \) are defined by (18) and (33).

Using (20) and (21) we can prove in the same way as in ([6], Ch.10, Th.3) that

\[ \sup_{\alpha \in E_2 \setminus \{1\}} |S(\alpha)| \ll \frac{N}{(\log N)^{B/2-4}}. \quad (48) \]

According to Kumchev ([7], Theorem 2) we have that

\[ \sup_{\alpha \in E_2 \setminus \{1\}} |\Sigma(\alpha)| \ll N^{1-\varepsilon}. \quad (49) \]

Bearing in mind (47)–(49) we get

\[ \sup_{\alpha \in E_2 \setminus \{1\}} |S_c(\alpha)| \ll \frac{N}{(\log N)^{B/2-4}}. \quad (50) \]

From (18) after straightforward computations we find

\[ \int_0^1 |S(\alpha)|^2 d\alpha \ll N \log N. \quad (51) \]

On the other hand, from (17) and (45) we obtain

\[
\int_0^1 |K(\alpha)|^2 d\alpha = \sum_{m_1, m_2 < D} \sum_{2|m_1, 2|m_2} \chi(j_1) \chi(j_2) \\
\times \int_0^1 S_{4m_1, 1+ j_1 m_1; J_m}(\alpha) S_{4m_2, 1+ j_2 m_2; J_m}(-\alpha) d\alpha \\
= \sum_{m_1, m_2 < D} \sum_{2|m_1, 2|m_2} \chi(j_1) \chi(j_2) \\
\times \sum_{p_1 \in J_m, i=1,2} \log p_1 \log p_2 \int_0^1 e(\alpha(p_1 - p_2)) d\alpha \\
\ll (\log N)^2 \sum_{m < D} \sum_{2|m} \sum_{p \in J_m, p \equiv 1 + jm (4m)} (\log p)^2 \\
\ll (\log N)^2 \sum_{m < D} \sum_{2|m} \sum_{p \in J_m, p \equiv 1 + jm (4m)} 1 \\
\ll N(\log N)^2 \sum_{m < D} \frac{1}{m} \\
\ll N \log^3 N. \quad (52)
\]
Thus from (46), (50) – (52) it follows
\[
\Gamma_3^{(2)}(N) \ll \frac{N^2}{(\log N)^{B/2-6}}.
\] (53)

Now let us consider \(\Gamma_3^{(1)}(N)\). From (42) and (44) for \(\nu = 1\) we get
\[
\Gamma_3^{(1)}(N) = \Gamma^* + \mathcal{O}(N^2(\log N)\Sigma_1) + \mathcal{O}(\tau^2(\log N)\Sigma_2)
+ \mathcal{O}(NQ^2(\log N)^2\Sigma_3) + \mathcal{O}(N^2e^{-c_0\sqrt{\log N}}\Sigma_4),
\] (54)

where
\[
\Gamma^* = \gamma \left( \sum_{m_1 + m_2 + m_3 = N \atop m_1 \in J_m} 1 \right) \sum_{m < D \atop 2|m} \frac{1}{\varphi(4m)} \sum_{j = \pm 1} \chi(j) \mathfrak{G}_{4m,1+jm}(N),
\]
\[
\Sigma_1 = \sum_{m < D \atop q > Q} (4m, q) \log q \quad mq^2,
\]
\[
\Sigma_2 = \sum_{m < D \atop q \leq Q} \frac{q}{[4m, q]},
\]
\[
\Sigma_3 = \sum_{m < D \atop q \leq Q} \Delta(N, [4m, q]) \quad q^2,
\]
\[
\Sigma_4 = \sum_{m < D} \frac{1}{m}.
\]

From the definition (5) it follows that \(\mathfrak{G}_{4m,1+jm}(N)\) does not depend on \(j\). Then we have
\[
\sum_{j = \pm 1} \chi(j) \mathfrak{G}_{4m,1+jm}(N) = 0
\]
and that leads to
\[
\Gamma^* = 0.
\] (55)

Arguing as in [13] and using Bombieri–Vinogradov’s theorem we find the following estimates
\[
\Sigma_1 \ll \frac{\log^3 N}{Q}, \quad \Sigma_2 \ll Q \log^2 N,
\] (56)
\[
\Sigma_3 \ll \frac{N}{(\log N)^{A-B-5}}, \quad \Sigma_4 \ll \log N.
\] (57)

Bearing in mind (21), (54) – (57) we obtain
\[
\Gamma_3^{(1)}(N) \ll \frac{N^2}{(\log N)^{B-4}}.
\] (58)

Now from (43), (53) and (58) we find
\[
\Gamma_3(N) \ll \frac{N^2}{(\log N)^{B/2-6}}.
\] (59)
6 Upper bound for $\Gamma_2(N)$

Consider the sum $\Gamma_2(N)$ defined by (13). We denote by $\mathcal{F}$ the set of all primes $p \leq N$ such that $p - 1$ has a divisor belongs to the interval $(D, N/D)$. Using the inequality $uv \leq u^2 + v^2$ and taking into account the symmetry with respect to $d$ and $t$ we get

$$\Gamma_2(N)^2 \ll (\log N)^6 N^{2-2\gamma} \sum_{\substack{p_1+p_2+p_3=N \\
p_4+p_5+p_6=N \\
p_2=[n_1],p_5=[n_2]}} \left| \sum_{D<d<N/D} \chi(d) \right| \sum_{D<d<N/D} \chi(t),$$

$$\ll (\log N)^6 N^{2-2\gamma} \sum_{\substack{p_1+p_2+p_3=N \\
p_4+p_5+p_6=N \\
p_2=[n_1],p_5=[n_2] \text{ or } p_4 \in \mathcal{F}}} \left| \sum_{D<d<N/D} \chi(d) \right|^2.$$  \hspace{1cm} (60)

Further, we use that if $n$ is a natural such that $n \leq N$, then the number of solutions of the equation $p_1 + p_2 = n$ in primes $p_1, p_2 \leq N$ such that $p_1 = \lceil m^{1/\gamma} \rceil$ is $O(N^\gamma (\log N)^{-2} \log \log N)$, i.e.

$$\# \{p_1 : p_1 + p_2 = n, p_1 = \lceil m^{1/\gamma} \rceil, n \leq N \} \ll \frac{N^\gamma \log \log N}{\log^2 N}.$$ \hspace{1cm} (61)

This follows for example from ([3], Ch.2, Th.2.4). Thus the summands in the sum (60) for which $p_1 = p_4$ can be estimated with $O(N^3 + \varepsilon)$.

Therefore

$$\Gamma_2(N)^2 \ll (\log N)^6 N^{2-2\gamma} \Sigma_1 + N^{3+\varepsilon},$$

where

$$\Sigma_1 = \sum_{p_1 \leq N} \left| \sum_{D<d<N/D} \chi(d) \right|^2 \sum_{p_4 \leq N} \sum_{p_2 \leq N} \sum_{p_5 \leq N} 1.$$ \hspace{1cm} (62)

We use again (61) and find

$$\Sigma_1 \ll \frac{N^{2\gamma}}{\log^4 N} (\log \log N)^2 \Sigma_2 \Sigma_3,$$ \hspace{1cm} (63)

where

$$\Sigma_2 = \sum_{p \leq N} \left| \sum_{D<d<N/D} \chi(d) \right|^2, \quad \Sigma_3 = \sum_{p \leq N} 1.$$ \hspace{1cm} (64)

Arguing as in ([4], Ch.5) we find

$$\Sigma_2 \ll \frac{N(\log \log N)^7}{\log N}, \quad \Sigma_3 \ll \frac{N(\log \log N)^3}{(\log N)^{1+2\theta_0}}.$$ \hspace{1cm} (65)

where $\theta_0$ is denoted by (4).

From (62) – (64) it follows

$$\Gamma_2(N) \ll N^2 (\log N)^{-6 \theta_0} (\log \log N)^6.$$ \hspace{1cm} (65)
7 Asymptotic formula for $\Gamma_1(N)$

In this section our argument is a modification of Tolev’s [14] argument. Consider the sum $\Gamma_1(N)$. From (12), (15) and (22) we get

$$\Gamma_1(N) = \Gamma_1^{(1)}(N) + \Gamma_1^{(2)}(N),$$

where

$$\Gamma_1^{(1)}(N) = \sum_{d \leq D} \chi(d) I_{d,1}^{(1)}(N),$$

$$\Gamma_1^{(2)}(N) = \sum_{d \leq D} \chi(d) I_{d,1}^{(2)}(N).$$

We estimate the sum $\Gamma_1^{(2)}(N)$ by the same way as the sum $\Gamma_3^{(2)}(N)$ and obtain

$$\Gamma_1^{(2)}(N) \ll \frac{N^2}{(\log N)^{B/2 - 6}},$$

Now we consider $\Gamma_1^{(1)}(N)$. We use the formula (42) for $J = [1, N]$. The error term is estimated by the same way as for $\Gamma_3^{(1)}(N)$. We have

$$\Gamma_1^{(1)}(N) = \frac{\gamma}{2} \mathcal{S}(N) N^2 \sum_{d \leq D} \chi(d) \mathcal{S}_{d,1}^*(N) \phi(d) + O\left(\frac{N^2}{(\log X)^{B-3}}\right),$$

where $\mathcal{S}(N)$ is defined by (6) and

$$\mathcal{S}_{d,1}^*(N) = \prod_{p | d, p \leq N} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \prod_{p \mid d, p \leq N-1} \left(1 - \frac{1}{(p-1)^2}\right) \times \prod_{p | d, p \leq N} \left(1 + \frac{1}{(p-1)^3}\right)^{-1} \prod_{p \mid d, p \leq N-1} \left(1 + \frac{1}{p-1}\right);$$

Denote

$$\Sigma = \sum_{d \leq D} f(d), \quad f(d) = \frac{\chi(d) \mathcal{S}_{d,1}^*(N)}{\phi(d)}.$$ 

We have

$$f(d) \ll d^{-1} (\log \log(10d))^2$$

with absolute constant in the Vinogradov’s symbol. Hence the corresponding Dirichlet series

$$F(s) = \sum_{d=1}^{\infty} \frac{f(d)}{d^s}$$

is absolutely convergent in $Re(s) > 0$. On the other hand, $f(d)$ is multiplicative with respect to $d$ and applying Euler’s identity we find

$$F(s) = \prod_p T(p, s), \quad T(p, s) = 1 + \sum_{l=1}^{\infty} f(p^l) p^{-ls}.$$
From (69), (70) and (72) we establish that
\[
T(p, s) = \left(1 - \frac{\chi(p)}{p^{s+1}}\right)^{-1} \left(1 + \frac{\chi(p)}{p^{s+1}} E_d(p)\right),
\]
where
\[
E_d(p) = \begin{cases} 
(p - 3)(p^2 - 3p + 3)^{-1} & \text{if } p \nmid N(N - 1), \\
(p - 1)^{-1} & \text{if } p \mid N, \\
(2p - 3)(p^2 - 3p + 3)^{-1} & \text{if } p \mid N - 1.
\end{cases}
\]

Hence we find
\[
F(s) = L(s + 1, \chi)N(s),
\]
where \(L(s + 1, \chi)\) is Dirichlet series corresponding to the character \(\chi\) and
\[
N(s) = \prod_{p \nmid N(N - 1)} \left(1 + \frac{\chi(p)}{p^{s+1}(p^2 - 3p + 3)}\right) \prod_{p \mid N} \left(1 + \frac{\chi(p)}{p^{s+1}(p - 1)}\right) \prod_{p \mid N - 1} \left(1 + \frac{\chi(p)}{p^{s+1}(p^2 - 3p + 3)}\right).
\]

From the properties of the \(L\)-functions it follows that \(F(s)\) has an analytic continuation to \(\text{Re}(s) > -1\). It is well known that
\[
L(s + 1, \chi) \ll 1 + |\text{Im}(s)|^{1/6} \text{ for } \text{Re}(s) \geq -\frac{1}{2}.
\]

Moreover,
\[
N(s) \ll 1.
\]

Using (73), (75) and (76) we get
\[
F(s) \ll N^{1/6} \text{ for } \text{Re}(s) \geq -\frac{1}{2}, \ |\text{Im}(s)| \leq N.
\]

We apply Perron’s formula given at Tenenbaum ([11], Chapter II.2) and also (71) to obtain
\[
\Sigma = \frac{1}{2\pi i} \int_{c-iN}^{c+iN} F(s) \frac{D^s}{s} ds + \mathcal{O}\left(\sum_{t=1}^{\infty} D^{\varepsilon} \log \log (10t) \left(1 + N \left|\log \frac{D}{t}\right|\right)\right),
\]
where \(\varepsilon = 1/10\). It is easy to see that the error term above is \(\mathcal{O}\left(N^{-1/20}\right)\).

Applying the residue theorem we see that the main term in (78) is equal to
\[
F(0) + \frac{1}{2\pi i} \left(\int_{-1/2-iN}^{1/2-iN} + \int_{1/2-iN}^{-1/2+iN} \right) F(s) \frac{D^s}{s} ds.
\]

From (77) it follows that the contribution from the above integrals is \(\mathcal{O}\left(N^{-1/20}\right)\).

Hence
\[
\Sigma = F(0) + \mathcal{O}\left(N^{-1/20}\right).
\]
Using (73) we get
\[ F(0) = \frac{\pi}{4} N(0). \] (80)

Bearing in mind (68), (70), (74), (79) and (80) we find a new expression for \( \Gamma_1^{(1)}(N) \)
\[ \Gamma_1^{(1)}(N) = \frac{\gamma}{8} \mathcal{S}_\Gamma(N) N^2 + \mathcal{O}\left(\frac{N^2}{(\log N)^{B-4}}\right), \] (81)
where \( \mathcal{S}_\Gamma \) is defined by (7).

From (66), (67) and (81) we obtain
\[ \Gamma_1(N) = \frac{\gamma}{8} \mathcal{S}_\Gamma(N) N^2 + \mathcal{O}\left(\frac{N^2}{(\log N)^{B/2-6}}\right). \] (82)

8 Proof of the Theorem

Therefore using (11), (59), (65) and (82) we find
\[ \Gamma(N) = \frac{\gamma}{2} \mathcal{S}_\Gamma(N) N^2 + \mathcal{O}\left(\frac{N^2}{(\log N)^{B-6}}\right). \]
This implies that \( \Gamma(N) \to \infty \) as \( N \to \infty \).

The Theorem is proved.

References


