The system of prime coordinates assigned to the positive integers

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Abstract: Due to the fundamental theorem of number theory, the positive integers may be represented by vectors whose components are the unique corresponding powers of the prime numbers. Taking the prime numbers as coordinates, to each positive integer we assign a prime vector, whose components are the powers of the prime factors of this integer. The geometry of this system of prime coordinates of the positive integers is discussed. It is shown that the prime components assigned to the sequence of positive integers change in a strictly deterministic way and the parallel generating system is presented. Gödel's prime vectors assigned to formal logical formulas are analyzed.

Keywords: Prime coordinates, Prime vectors, Regular structure of the prime coordinates, Parallel generating system, Gödel's prime vectors.

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1 Introduction

Every positive integer larger than 1 is the unique product of a finite number of primes or powers of primes. Taking the prime numbers as coordinates, to each positive integer we assign a prime vector, whose components are the powers of the prime factors of this integer. Taking into account the system of vectors whose components are powers of all the prime numbers, to each positive integer it corresponds a unique such a prime vector which has no more than a finite positive components. The integer 1 corresponds to the origin of the system of prime coordinates. Thus, we have:

$$1 = (0, 0, 0, 0, 0, \ldots), \; 2 = (1, 0, 0, 0, 0, \ldots), \; 3 = (0, 1, 0, 0, 0, \ldots), \; 4 = (2, 0, 0, 0, 0, \ldots), \ldots$$

Equivalently, we can mention explicitly only the positive exponents of the corresponding prime factors and assume that the exponents of all the other primes are equal to 0. Thus,

$$2 = [1; 2], \ 3 = [1,3], \ 4 = [2; 2], \ 5 = [1, 5], \ 6 = [1,1; 2,3], \ \ldots$$
$$\ldots \ 10000 = [4,4; 2,5], \ \ldots \ \ldots, \ 63992 = [3,1,1; 2,19,421], \ \ldots$$
$$\ldots 78936 = [3,1,1,11; 2.3.11,13,23], \ \ldots \ 97823 = [1,1; 11,8893], \ldots$$

The geometry of the vectors with prime coordinates is strange but relevant. Denoting by p_k the k-th prime number, then $p_k(n)$ is the k-th component of the positive integer n, which is the power of the prime number p_k in the decomposition of n in prime factors. The norm of the positive integer n is:

$$\| n \| = \sqrt{\sum\nolimits_{k=1}^{\infty} (p_k(n))^2}.$$

As every positive integer is the product of a finite number of primes, the above series is a finite sum. The numerical value of the norm of the positive integer n essentially depends on the multiplicity of the prime factors of n. Thus, in spite of the fact that 6 and 97823 are very different integers, they have the same norm, namely $\|6\| = \|97823\| = \sqrt{2}$. Because both numbers have only two prime factors of multiplicity 1. Also, $\|78936\| = \sqrt{13}$ and $\|4| = 2$. Obviously, each prime number has the norm equal to 1.

The scalar product of the positive integers m and n is:

$$< m \mid n > = \sum\nolimits_{k=1}^{\infty} p_k(m) p_k(n).$$

As every positive integer is the product of a finite number of primes, the above series is a finite sum.

The scalar product essentially depends on the multiplicity of the common prime factors.

Obviously, two different primes are orthogonal but there are positive integers which are orthogonal without being primes, such as $< 63922 \mid 97823 > \ = 0$ for instance, because the two integers have no common prime factors. Finally, the usual product of positive integers corresponds to the sum of the corresponding prime vectors. Thus, the prime vector corresponding to $12 \times 66 = 792$ is:

$$[2,1; 2,3] + [1,1,1; 2,3,11] = [3,2,1; 2,3,11].$$

In the next two sections we discuss the regular way in which the components of the prime vectors are successively generated.

2   The periodic structure of the components of the prime vectors

Let p be an arbitrary prime number. It corresponds to an arbitrary coordinate of the system of prime vectors discussed above. For each positive integer r we define the periodic function:

$$f_{p,r}(n) = \begin{cases} 1, & \text{if } n = mp^r \ (m = 1, 2, 3, \ldots) \\ 0, & \text{otherwise} \end{cases},$$

which assigns the value 1 to all multiples of p^r and the value 0 to the other positive integers. The p-th prime coordinate of the positive integer n or, equivalently, the power of the prime p in the factorization of n, is equal to the sum of these periodic functions at n, i.e.

$$p(n) = \sum_{r=1}^{\infty} f_{p,r}(n). \tag{1}$$

As the positive integer n has only a finite number of prime factors, the series in (1) is in fact a finite sum. For instance,

$$2(24) = f_{2,1}(24) + f_{2,2}(24) + f_{2,3}(24) = 1 + 1 + 1 = 3,$$

$$3(24) = f_{3,1}(24) = 1.$$

All the other values of the periodic functions (1) corresponding to $n = 24$ are equal to 0 and we have $24 = [3,1; 2,3]$.

3 Parallel system assigning integer powers of primes to integers

Let S be a system which assigns powers of the prime numbers to the positive integer numbers larger than 1. For every prime p and every positive integer r there is a component $X_{p,r}$ of the system S which assigns the value 1 (success) to the integers $n = mp^r$ ($m = 1, 2, 3, \ldots$) and 0 (failure) to the other positive integers. The structure function of the function $f_{p,r}$ is the component $X_{p,r}$ defined in the previous section. The component $X_{p,r}$ functions properly when it assigns the value 1 to a positive integer and fails to function properly when it assigns the value 0 to a positive integer. Taking the positive integers as being successive time instants, the structure function $f_{p,r}$ is periodic, with the period p^r, for the component $X_{p,r}$ the probability of success is p^{-r} and the probability of a failure is $1 - p^{-r}$. Reliability of the component $X_{p,r}$ is the probability that this component function properly, namely:

$$R_{p,r} = P(f_{p,r} = 1) = p^{-r}. \tag{2}$$

The components $X_{p,r}$ of the system S are joined in parallel. A parallel system functions properly if and only if all its components function properly. It fails to function properly if at least one of its components fails to function properly. The structure function of the parallel system S is:

$$f(n) = \max_{p,r}\{f_{p,r}(n)\} = 1 - \prod_{p,r}[1 - f_{p,r}(n)], \tag{3}$$

As every positive integer has only a finite number of prime factors, for each integer n only a finite number of elements of the set $\{f_{p,r}(n)\}$ are equal to 1, the rest of them being equal to 0. Using (2), the reliability of the parallel system S is:

$$R = \max_{p,r}\{R_{p,r}\} = 1 - \prod_{p,r}(1 - R_{p,r}) = 1 - \prod_{p,r}(1 - p^{-r}). \tag{4}$$

By proper functioning, the system S assigns integer powers to the prime factors of the positive integer numbers. But, due to the fundamental theorem of number theory, every positive integer larger than 1 is the product of a finite number of primes or positive integer powers of primes. Therefore, the system S functions properly covering all positive integers larger than 1 with finite products of primes or powers of primes.

Consequently, the reliability of the parallel system S is maximum and is equal to $R - 1$. As a corollary, the probability that the system S fails to assign positive integer powers to the positive integer numbers larger than 1 is equal to $1 - R = 0$, which, according to (4), states that:

$$\prod_{p,r}(1 - p^{-r}) = 0, \tag{5}$$

which is a variant of Euler's formula for the product of the reciprocal of primes.

4   Gödel's prime vectors

Using an ingenious numbering system, entirely based on the prime numbers, Gödel arithmetized the mathematical logic and showed that any system of arithmetic axiomatic system contains a non-demonstrable, or non-provable statement. In this section, using Kurt Gödel methodology [2], the prime coordinates and Alonzo Church's version of the liar paradox [1], a simple non-provable statement is given.

Using the version of Gödel theory given in [5], we assign the integers 1, 2, 3, 4, 5, 6, 7, to the elementary constant signs: of Gödel ~ (non), $\vee$ (or), $\supset$ (implies, or, equivalently, if $\cdots$ then $\cdots$), $\exists$ (there is), = (equal), 0 (zero), s (the immediate successor of). The punctuation marks (and) have the Gödel numbers 8 and 9, respectively. A numerical variable, like x or y, gets a prime larger than 10 as the Gödel number. A sentential variable gets as the Gödel number the square of a prime number larger than 10. Let p and q be two sentential variables with the 11^2 and 13^2, respectively. The logical sentence:

$$q(p) = {\sim}p(p). \tag{6}$$

corresponds to the prime vector:

$$[13^2, 8, 11^2, 9, 5, 1, 11^2, 8, 11^2, 9, 2, 3, 5, 7, 11, 13, 17, 10, 23, 29],$$

whose components are the Gödel numbers of the signs of the expression (6). The Gödel number of the entire given logical expression (6) is:

$$n = 2^{13^2} \times 3^8 \times 5^{11^2} \times 7^9 \times 11^5 \times 13^1 \times 17^{11^2} \times 19^8 \times 23^{11^2} \times 29^9.$$

The number n is a label, or tag, for the logical expression (6).

If x is the Gödel number of a sequence of symbols, y is a numerical variable whose value is the Gödel numbers of a variable, and z is a numerical variable whose value is the Gödel number of a sentence or variable, sub(x, y, z) is the sequence labeled by x in which the variable labeled by y is replaced by the sequence or variable labeled by z. Thus, sub$(n, 11^2, 13^2)$ is the expression:

$$q(q) = {\sim}q(q). \tag{7}$$

Let H be the set of axioms and $P(H)$ be the class of the subsets of H. The following statement in non-provable:

$$(h)\ h \sim\ \supset \text{sub}(n, 11^2, 13^2), \tag{8}$$

which says that: for every set of axioms, we cannot prove the relation (7). Indeed, if (8) is true, then the system of axioms H is not consistent. If (8) is false, then:

$$(\exists h)\ h \supset \text{sub}(n, 11^2, 13^2),$$

which says that there is a set of axioms h that implies (7) which is a contradiction. Therefore, if (8) is provable then either the system of axioms is inconsistent or it contains a contradiction, Thus, (8) is not provable.

5   Conclusion

Taking the prime numbers as coordinates, to every positive integer n corresponds a vector, called the prime vector assigned to n, whose components are the powers of those primes that divide n. This is called the prime vector assigned to n. The geometry of these prime vectors assigned to the positive integers is unusual but relevant. The scalar product of the prime vectors corresponding to two positive integers m and n, reveals the multiplicity of the common prime factors of m and n, and the norm of the prime vector corresponding to the positive integer n reflects the multiplicity of the prime divisors of n. The components of the prime vectors assigned to the positive integers display a clear regularity, as shown by (1), each such component being the sum of the corresponding values of periodic functions. Finally, due to the fundamental theorem of number theory, the parallel system of assigning positive powers of primes to the positive integers larger than 1 proves to have a maximum reliability $R = 1$, which is equivalent to a strong version (5) of Euler's formula for the product of reciprocal primes. Gödel has represented the formal logical statements as prime vectors. Using his methodology and Church's version of the liar paradox, a simple non provable statement (8) is given.

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