The arrowhead-Pell-random-type sequences

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Abstract: In this paper, we define the arrowhead-Pell-random-type sequence and then we ob-
tain the generating function and the generating matrix of the sequence. Also, we derive the
permanental, determinantal, combinatorial and exponential representations and the sums of the
arrowhead-Pell-random-type numbers using the generating function and the generating matrix of
the sequence.

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1 Introduction

In [13], Kılıç and Tasci defined \( k \)-sequences of the generalized order-\( k \) Pell numbers as shown:

\[
P_n^i = 2P_{n-1}^i + P_{n-2}^i + \cdots + P_{n-k}^i
\]

for \( n > 0 \) and \( 1 \leq i \leq k \), with initial conditions

\[
P_n^i = \begin{cases} 
1 & \quad \text{if } n = 1 - i, \\
0 & \quad \text{otherwise,}
\end{cases} \quad 1 - k \leq n \leq 0,
\]

where \( P_n^i \) is the \( n \)-th term of the \( i \)-th sequence.

It is clear that the characteristic polynomial of the generalized order-\( k \) Pell sequence is as
follows:

\[
P(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - 1.
\]
In [1], Aküzüm et al. defined the arrowhead-Pell sequence for \( n \geq 1 \) as follows:

\[
a_{k+1} (n + k + 1) = a_{k+1} (n + k) - 2a_{k+1} (n + k - 1) - a_{k+1} (n + k - 2) - \cdots - a_{k+1} (n)
\]

with integer constants \( a_{k+1} (1) = \cdots = a_{k+1} (k) = 0 \) and \( a_{k+1} (k + 1) = 1 \), where \( k \geq 2 \).

Shannon and Horadam [17] also developed arrowhead curves in the context of recursive sequences.

Hofstadter’s integer sequences defined [10] by

\[
h_n = h_{n-1} - h_{n-2} + h_{n-3} - h_{n-4}
\]

where \( h_1 = h_2 = 1 \).

The random Fibonacci sequences defined [6] by the random recurrence \( x_1 = 1, x_2 = 2 \) and for \( n > 2 \), \( x_n = \pm x_{n-1} \pm x_{n-2} \), where each \( \pm \) sign is independent and either + or − with probability \( 1/2 \).

Atanassov et al. [2] have, to some extent, systematized aspects of these sequences through pulsed sequences.

Suppose that the \( (n + k) \)th term of a sequence is defined recursively by a linear combination of the preceding \( k \) terms:

\[
a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}
\]

where \( c_0, c_1, \ldots, c_{k-1} \) are real constants. In [11], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

\[
A = [a_{i,j}]_{k \times k} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1}
\end{bmatrix}
\]

Then by an inductive argument he obtained that

\[
A^n \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{k-1}
\end{bmatrix} = \begin{bmatrix}
a_n \\
a_{n+1} \\
\vdots \\
a_{n+k-1}
\end{bmatrix}
\]

for \( n \geq 0 \).

Number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper have been studied by many authors [4, 7, 8, 9, 12, 16, 18, 19, 20, 21, 22, 23]. In this paper, we define a new sequence which is called the arrowhead-Pell-random-type sequence. Then we give relationships among the arrowhead-Pell-random-type numbers and the permanents and the determinants of certain matrices which are produced by using the
generating matrix of the arrowhead-Pell-random-type sequence. Also, we obtain the combinatorial representations, the exponential representation and the sums of the arrowhead-Pell-random-type numbers by the aid of the generating function and the generating matrix of the arrowhead-Pell-random-type sequence.

2 The arrowhead-Pell-random-type sequence

We now define the arrowhead-Pell-random-type sequence by the following recurrence relations for \( n \geq u \)

\[
a_k^{u} (n + k + 1) = a_k^{u} (n + k - u) - 2a_k^{u} (n + k - u - 1) - a_k^{u} (n + k - u - 2) - \cdots - a_k^{u} (n - u)
\]

(2.1)

with initial conditions \( a_k^{u} (0) = \cdots = a_k^{u} (u + k - 1) = 0 \) and \( a_k^{u} (u + k) = 1 \), where \( 1 \leq u \leq k + 1 \) and \( k \geq 2 \).

By (2.1), we can write a generating matrix for the arrowhead-Pell-random-type sequence as follows:

\[
A_{k,u} = \begin{bmatrix}
0 & \cdots & 0 & 1 & -2 & -1 & \cdots & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\( (u + 1) \times (u + k + 1) \)

The companion matrix \( A_{k,u} \) is called the arrowhead-Pell-random-type matrix. It is clear that

\[
(A^{k,u})^\alpha = \begin{bmatrix}
a_k^{u} (u + k) \\
a_k^{u} (u + k - 1) \\
\vdots \\
a_k^{u} (0)
\end{bmatrix}

= \begin{bmatrix}
a_k^{u} (\alpha + u + k) \\
a_k^{u} (\alpha + u + k - 1) \\
\vdots \\
a_k^{u} (\alpha)
\end{bmatrix}
\]

for \( \alpha \geq u \). Let \( a_k^{u} (\alpha) \) be denoted by \( a_{k+1}^{u,\alpha} \). By induction on \( \alpha \), we derive that
such that \( \alpha > 1 \) is called the contraction in the \( k \)th column from which it is clear that \( \det (\mathbf{A}) = (u+1) \). Where \((\mathbf{A}^k,u)\) is a \((u+k+1)\times(k-2)\) matrix as follows:

\[
\begin{pmatrix}
\begin{array}{cccccc}
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\end{array}
\end{pmatrix}
\]

for \( k \geq 2 \). Where \((\mathbf{A}^k,u)\) is a \((u+k+1)\times(k-2)\) matrix as follows:

\[
\begin{pmatrix}
\begin{array}{cccccc}
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\end{array}
\end{pmatrix}
\]

such that

\[
\begin{pmatrix}
\begin{array}{cccccc}
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\end{array}
\end{pmatrix}
\]

from which it is clear that \( \det (\mathbf{A}^k,u) = (-1)^{(u+k+1)} \).

Now we consider the permanental representations of the arrowhead-Pell-random-type sequence.

**Definition 2.1.** A \( u \times v \) real matrix \( \mathbf{M} = [m_{ij}] \) is called a contractible matrix in the \( k \)th column (resp. row) if the \( k \)th column (resp. row) contains exactly two non-zero entries.

Suppose that \( x_1, x_2, \ldots, x_u \) are row vectors of the matrix \( \mathbf{M} \). If \( \mathbf{M} \) is contractible in the \( k \)th column such that \( m_{i,k} \neq 0, m_{j,k} \neq 0 \) and \( i \neq j \), then the \((u-1) \times (v-1)\) matrix \( \mathbf{M}_{i,j:k} \) obtained from \( \mathbf{M} \) by replacing the \( i \)th row with \( m_{i,k} x_j + m_{j,k} x_i \) and deleting the \( j \)th row. The \( k \)th column is called the contraction in the \( k \)th column relative to the \( i \)th row and the \( j \)th row.

In [3], Brualdi and Gibson obtained that \( \text{per} (\mathbf{M}) = \text{per} (\mathbf{N}) \) if \( \mathbf{M} \) is a real matrix of order \( \alpha > 1 \) and \( \mathbf{N} \) is a contraction of \( \mathbf{M} \).

Let \( m \geq u + k + 1 \) be a positive integer and suppose that \( H^{k,u} (m) = \left[h^{m,k,u}_{i,j}\right] \) is the \( m \times m \) super-diagonal matrix, defined by:

\[
(\mathbf{A}^{k,u})^\alpha = \begin{pmatrix}
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\end{pmatrix}
\]

\[
(M^{k,u})^\alpha = \begin{pmatrix}
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\end{pmatrix}
\]

\[
(M^{*k,u})^\alpha = \begin{pmatrix}
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{k+1} & \alpha_{k+1} & \cdots & \alpha_{k+1} \\
\end{pmatrix}
\]
That is,

\[ h_{i,j}^{m,k,u} = \begin{cases} 
1 & \text{if } i = r \text{ and } j = r + u \text{ for } 1 \leq r \leq m - u - k \\
-1 & \text{if } i = r + 1 \text{ and } j = r \text{ for } 1 \leq r \leq m - 1, \\
-1 & \text{if } i = r \text{ and } j = r + u + 2 \text{ for } 1 \leq r \leq m - u - k, \\
-1 & \text{if } i = r \text{ and } j = r + u + 3 \text{ for } 1 \leq r \leq m - u - k, \\
\vdots & \\
-2 & \text{if } i = r \text{ and } j = r + u + k \text{ for } 1 \leq r \leq m - u - k, \\
0 & \text{otherwise.} 
\end{cases} \]

That is,

\[
\begin{pmatrix}
(u+1)th & \downarrow & (u+k+1)th \\
0 & \ldots & 0 & 1 & -2 & -1 & \ldots & -1 & 0 & 0 & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 & 1 & -2 & -1 & \ldots & -1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 & 1 & -2 & -1 & \ldots & -1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 & -2 & -1 & \ldots & -1 & 0 \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 & -2 & -1 & \ldots & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 & -2 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 \\
\end{pmatrix}
\]

Then we have the following Theorem.

**Theorem 2.1.** For \( m \geq u + k + 1 \) and \( k \geq 2 \),

\[
\per H^{k,u} (m) = a_{k+1}^u (m + u + k).
\]

**Proof.** Let the equation hold for \( m \geq u + k + 1 \), then we show that the equation holds for \( m + 1 \). If we expand the \( \per H^{k,u} (m) \) by the Laplace expansion of permanent with respect to the first row, then we obtain

\[
\per H^{k,u} (m + 1) = \per H^{k,u} (m - u) - 2 \per H^{k,u} (m - u - 1) - \per H^{k,u} (m - u - 2) - \cdots - \per H^{k,u} (m - u - k).
\]

Since

\[
\per H^{k,u} (m - u) = a_{k+1}^u (m + k),
\]

\[
\per H^{k,u} (m - u - 1) = a_{k+1}^u (m + k - 1),
\]

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\[ \text{per} H^{k,u} (m - u - 2) = a_{k+1}^u (m + k - 2), \ldots, \text{per} H^{k,u} (m - u - k) = a_{k+1}^u (m), \]

we easily obtain that \( \text{per} H^{k,u} (m + 1) = a_{k+1}^u (m + u + k + 1) \).

So the proof is complete.

Let \( m > u + k + 1 \) such that \( k \geq 2 \) and let \( L^{k,u} (m) = \left[ l_{i,j}^{m,k,u} \right] \) be the \( m \times m \) matrix, defined by

\[
l_{i,j}^{m,k,u} = \begin{cases} 
1 & \text{if } i = r \text{ and } j = r + u \text{ for } 1 \leq r \leq m - u - k \\
1 & \text{and } i = r + 1 \text{ and } j = r \text{ for } 1 \leq r \leq m - 1, \\
i = r \text{ and } j = r + u + 2 \text{ for } 1 \leq r \leq m - u - k, \\
i = r \text{ and } j = r + u + 3 \text{ for } 1 \leq r \leq m - u - k, \\
\vdots\\n-2 & \text{if } i = r \text{ and } j = r + u \text{ for } 1 \leq r \leq m - u - k, \\
0 & \text{otherwise.}
\end{cases}
\]

That is,

\[
L^{k,u} (m) = \begin{bmatrix}
0 & \cdots & 0 & 1 & -2 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 1 & -2 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 1 & -2 & -1 & \cdots & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & -2 & -1 & \cdots & -1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & -2 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\
\end{bmatrix} \leftarrow (m-u-k) \text{ th.}
\]

Assume that the \( m \times m \) matrix \( K^{k,u} (m) = \left[ k_{i,j}^{m,k,u} \right] \) is defined by
Then we can give more general results by using other permanental representations than the above.

**Theorem 2.2.** Let $a_{k+1}^u(m)$ be the $m$th the arrowhead-Pell-random-type number for $k \geq 2$. Then

(i). For $m > u + k + 1$,

$$\text{per}L_{k,u}(m) = -a_{k+1}^u(m - 1).$$

(ii). For $m > u + k + 2$,

$$\text{per}K_{k,u}(m) = \sum_{i=1}^{m-2} a_{k+1}^u(i).$$

**Proof.** (i). Let the equation hold for $m > u + k + 1$, then we show that the equation holds for $m + 1$. If we expand the $\text{per}L_{k,u}(m)$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$\text{per}L_{k,u}(m + 1) = \text{per}L_{k,u}(m - u) - 2\text{per}L_{k,u}(m - u - 1) - \text{per}L_{k,u}(m - u - 2) - \cdots - \text{per}L_{k,u}(m - u - k).$$

Also, since

$$\text{per}L_{k,u}(m - u) = -a_{k+1}^u(m - u - 1),$$

$$\text{per}L_{k,u}(m - u - 1) = -a_{k+1}^u(m - u - 2),$$

$$\text{per}L_{k,u}(m - u - 2) = -a_{k+1}^u(m - u - 3), \ldots,$$

$$\text{per}L_{k,u}(m - u - k) = -a_{k+1}^u(m - u - k - 1),$$

it is clear that

$$\text{per}L_{k,u}(m + 1) = -a_{k+1}^u(m).$$

(ii). It is clear that expanding the $\text{per}K_{k,u}(m)$ by the Laplace expansion of permanent with respect to the first row, gives us

$$\text{per}K_{k,u}(m) = \text{per}K_{k,u}(m - 1) + \text{per}L_{k,u}(m - 1).$$

By induction on $m$, taking into consideration the result of Theorem 2.1 and part (i) in Theorem 2.2, the conclusion is easily seen. Q.E.D.
Let the notation $A \circ K$ denote the Hadamard product of $A$ and $K$. A matrix $A$ is called convertible if there is an $m \times m$ $(1, -1)$-matrix $K$ such that $\text{per} A = \det (A \circ K)$.

Let $m > u + k + 2$ and let $R$ be the $m \times m$ matrix, defined by

$$R = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & 1 & 1 & \cdots & 1 & 1 \\
1 & -1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cdots & 1 & -1 & 1 & 1 \\
1 & \cdots & 1 & 1 & -1 & 1 \\
\end{bmatrix}.$$  

It is easy to see that $\text{per} H^{k,u}(m) = \det (H^{k,u}(m) \circ R)$, $\text{per} L^{k,u}(m) = \det (L^{k,u}(m) \circ R)$ and $\text{per} K^{k,u}(m) = \det (K^{k,u}(m) \circ R)$ for $m > u + k + 2$. Then we have the following useful results.

**Corollary 2.3.** For $m > u + k + 2$,

$$\det (H^{k,u}(m) \circ R) = a_{k+1}^u (m + u + k),$$

$$\det (L^{k,u}(m) \circ R) = -a_{k+1}^u (m - 1)$$

and

$$\det (K^{k,u}(m) \circ R) = \sum_{i=1}^{m-2} a_{k+1}^u (i).$$

Let $C(c_1, c_2, \ldots, c_v)$ be a $v \times v$ companion matrix as follows:

$$C(c_1, c_2, \ldots, c_v) = \begin{bmatrix}
c_1 & c_2 & \cdots & c_v \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 \\
\end{bmatrix}.$$  

See [14, 15] for more information about the companion matrix.

**Theorem 2.4** (Chen and Louck [5]). The $(i, j)$ entry $c_{i,j}^{(n)}(c_1, c_2, \ldots, c_v)$ in matrix $C^n(c_1, c_2, \ldots, c_v)$ is given by the following formula:

$$c_{i,j}^{(n)}(c_1, c_2, \ldots, c_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} c_1^{t_1} \cdots c_v^{t_v} \quad (2.2)$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $\binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v}$ is a multinomial coefficient, and the coefficients in (2.2) are defined to be 1 if $n = i - j$.

Then we give the combinatorial representations for the arrowhead-Pell-random-type numbers.
Corollary 2.5. Let \( a_{k+1}^u (\alpha) \) be the \( \alpha \)th the arrowhead-Pell-random-type number for \( k \geq 2 \). Then

\[
(i) \quad a_{k+1}^u (\alpha) = \sum_{(t_1, t_2, \ldots, t_{u+k+1})} \left( t_1 + \cdots + t_{u+k+1} \right) \left( -2 \right)^{t_{u+2}} \left( -1 \right)^{t_{u+3} + t_{u+4} + \cdots + t_{u+k+1}} \]

where the summation is over nonnegative integers satisfying \( t_1 + 2t_2 + \cdots + (u + k + 1) t_{u+k+1} = \alpha - u - k \).

\[
(ii) \quad a_{k+1}^u (\alpha) = - \sum_{(t_1, t_2, \ldots, t_{u+k+1})} \frac{t_{u+k+1}}{t_1 + t_2 + \cdots + t_{u+k+1}} \times \left( t_1 + \cdots + t_{u+k+1} \right) \left( -2 \right)^{t_{u+2}} \left( -1 \right)^{t_{u+3} + t_{u+4} + \cdots + t_{u+k+1}} \]

where the summation is over nonnegative integers satisfying \( t_1 + 2t_2 + \cdots + (u + k + 1) t_{u+k+1} = \alpha + 1 \).

**Proof.** If we take \( i = u + k + 1, j = 1 \), \( c_1 = \cdots = c_u = 0, c_{u+1} = 1, c_{u+2} = -2, c_{u+3} = \cdots = c_{u+k+1} = -1 \) for the case \( (i) \) and \( i = u + k, j = u + k + 1, c_1 = \cdots = c_u = 0, c_{u+1} = 1, c_{u+2} = -2, c_{u+3} = \cdots = c_{u+k+1} = -1 \) for the case \( (ii) \). in Theorem 2.4, then the proof is immediately seen from \((M^k)^\alpha\). \( \square \)

It is easy to show that the generating function of the arrowhead-Pell-random-type sequence is as follows:

\[
g_{k,u} (x) = \frac{x^{u+k}}{1 - x^{u+1} + 2x^{u+2} + x^{u+3} + \cdots + x^{u+k+1}} \]

where \( k \geq 2 \).

Now we give an exponential representation for the arrowhead-Pell-random-type numbers by the aid of the generating function with the following Theorem.

**Theorem 2.6.** The arrowhead-Pell-random-type numbers have the following exponential representation:

\[
g_{k,u} (x) = x^{u+k} \exp \left( \sum_{i=1}^{\infty} \frac{(x^{u+1})^i}{i} \left( 1 - x - \cdots - x^k \right)^i \right),
\]

where \( k \geq 2 \).

**Proof.** Since

\[
\ln g_{k,u} (x) = \ln x^{u+k} - \ln \left( 1 - x^{u+1} + 2x^{u+2} + x^{u+3} + \cdots + x^{u+k+1} \right)
\]

and

\[
- \ln \left( 1 - x^{u+1} + 2x^{u+2} + x^{u+3} + \cdots + x^{u+k+1} \right) = - \left[ -x^{u+1} \left( 1 - 2x - x^2 - \cdots - x^k \right) - \right.
\]

\[
\frac{1}{2} \left( x^{u+1} \right)^2 \left( 1 - 2x - x^2 - \cdots - x^k \right)^2 - \cdots
\]

\[
- \frac{1}{n} \left( x^{u+1} \right)^n \left( 1 - 2x - x^2 - \cdots - x^k \right)^n - \cdots \]

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it is clear that
\[ \ln g^{k,u}(x) - \ln x^{u+k} = \ln \frac{g^{k,u}(x)}{x^{u+k}} = \sum_{i=1}^{\infty} \frac{(x^{u+1})^i}{i} \left(1 - 2x - x^2 - \ldots - x^k\right)^i. \]

Thus we have the conclusion. \(\square\)

Now we consider the sums of arrowhead-Pell-random-type numbers.

Let
\[ S_\alpha = \sum_{i=1}^{\alpha} a_{k+1}^u (i) \]
for \(\alpha > 1\) and \(k \geq 2\), and suppose that \(E^{u,k}\) is the \((u + k + 2) \times (u + k + 2)\) matrix such that
\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & A^{k,u}
\end{bmatrix}.
\]

Then it can be shown by induction that
\[
\left( E^{k,u} \right)^\alpha = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
S_{\alpha+u+k-1} & 0 & \cdots & 0 \\
S_{\alpha+u+k-2} & (A^{k,u})^\alpha & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
S_{\alpha-1} & \cdots & 0 & (A^{k,u})^\alpha
\end{bmatrix}.
\]

References


