Fibonacci and Lucas numbers
via the determinants of tridiagonal matrix

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Abstract: Applying the apparatus of triangular matrices, we proved new recurrence formulas
for the Fibonacci and Lucas numbers with even (odd) indices by tridiagonal determinants.

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1 Triangular matrix and parapermanents of triangular matrix

The functions of triangular matrices are widely used in algebra, combinatorics, number theory
and other branches of mathematics [9, 11, 12].

Definition 1.1. [11]. A triangular number table

\[
A_n = \begin{pmatrix}
    a_{11} & & \\
    a_{21} & a_{22} & \\
    \vdots & \vdots & \ddots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]  \hspace{1cm} (1)

is called a \( n \)th-order triangular matrix.

Note that a matrix (1) is not a triangular matrix in the usual sense of this term as it is not a square matrix.
The product $a_{ij} a_{i,j+1} \cdots a_{ii}$ is denoted by $\{a_{ij}\}$ and is called a \textit{factorial product} of the element $a_{ij}$.

\textbf{Definition 1.2.} [11]. The parapermanent $\text{pper}(A_n)$ of a triangular matrix (1) is the number

$$\text{pper}(A_n) \equiv \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{21} & a_{22} & \ldots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} = \sum_{r=1}^{n} \sum_{p_1, \ldots, p_r = n} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_r, n-s+1}\}, \quad (2)$$

where $p_1, p_2, \ldots, p_r$ are positive integers, $\{a_{ij}\}$ is the factorial product of the element $a_{ij}$.

\textbf{Example 1.3.} The parapermanent of a 4-th order matrix:

$$\text{pper}(A_4) = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{21} & a_{22} & a_{32} & a_{42} \\ a_{31} & a_{32} & a_{33} & a_{43} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{41}a_{42}a_{43}a_{44} + a_{31}a_{32}a_{33}a_{44} + a_{21}a_{22}a_{43}a_{44} + a_{21}a_{22}a_{33}a_{44} + a_{11}a_{42}a_{43}a_{44} + a_{11}a_{32}a_{33}a_{44} + a_{11}a_{22}a_{43}a_{44} + a_{11}a_{22}a_{33}a_{44}.$$

To each element $a_{ij}$ of a matrix (1) we associate the triangular table of elements of matrix $A_n$ that has $a_{ij}$ in the bottom left corner. We call this table a \textit{corner} of the matrix and denote it by $R_{ij}(A_n)$. Corner $R_{ij}(A_n)$ is a triangular matrix of order $(i-j+1)$, and it contains only elements $a_{rs}$ of matrix (1) whose indices satisfy the inequalities $j \leq s \leq r \leq i$.

\textbf{Theorem 1.4.} [11] (Decomposition of a parapermanent $\text{pper}(A_n)$ by elements of the last row). The following formula are valid:

$$\text{pper}(A_n) = \sum_{s=1}^{n} \{a_{ns}\} \text{pper}(R_{s-1,1}(A_n)), \quad (3)$$

where $\text{pper}(R_{0,1}(A_n)) \equiv 1$.

\textbf{Example 1.5.} Decomposition of a parapermanent $\text{pper}(A_4)$ by elements of the last row:

$$\text{pper}(A_4) = a_{44}\text{pper}(A_3) + a_{43}a_{44}\text{pper}(A_2) + a_{42}a_{43}a_{44}\text{pper}(A_1) + a_{41}a_{42}a_{43}a_{44}\text{pper}(A_0),$$

where $\text{pper}(A_1) = a_{11}$, $\text{pper}A_0 \equiv 1$.

R. Zatorsky and I. Lishchynskyy [10, 13] established connection between the paradeterminants and the lower Hessenberg determinants by formula

$$\text{pper}(A_n) = \begin{bmatrix} \{a_{11}\} & 1 & 0 & \ldots & 0 & 0 \\ -\{a_{21}\} & \{a_{22}\} & 1 & \ldots & 0 & 0 \\ -\{a_{31}\} & -\{a_{32}\} & \{a_{33}\} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\{a_{n-1,1}\} & -\{a_{n-1,2}\} & -\{a_{n-1,3}\} & \ldots & \{a_{n-1,n-1}\} & 1 \\ -\{a_{n1}\} & -\{a_{n2}\} & -\{a_{n3}\} & \ldots & -\{a_{n,n-1}\} & \{a_{nn}\} \end{bmatrix}, \quad (4)$$

where $\{a_{ij}\}$ is factorial product of the element $a_{ij}$. 

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A connection between the Horadam numbers with even (odd) indices and parapermanents

In [5] A. Horadam considered the sequence

\[ h_1 = p, \ h_2 = q, \ h_n = h_{n-1} + h_{n-2}, \ n \geq 3, \]

where \( p \) and \( q \) are arbitrary integer numbers. This sequence generalized the Fibonacci sequence:

\[ F_1 = 1, \ F_2 = 1, \ F_n = F_{n-1} + F_{n-2}, \ n \geq 3, \]

and the Lucas sequence:

\[ L_1 = 2, \ L_2 = 1, \ L_n = L_{n-1} + L_{n-2}, \ n \geq 3. \]

**Proposition 2.1.** The following formula is valid:

\[
h_{2n-1} = \begin{bmatrix}
p & \ h_2 & 1 \\
\ h_1 & 1 & \ h_4 \\
\ h_3 & 1 & \ h_6 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & \ h_{2n-4}^{\ h_{2n-5}} & 1 \\
0 & 0 & 0 & \cdots & 0 & \ h_{2n-2}^{\ h_{2n-5}} & 1
\end{bmatrix}.
\] (5)

**Proof.** Expanding the parapermanent (5) by elements of the last raw (see (3)), we have

\[ h_{2n-1} = 1 \cdot h_{2n-3} + \ h_{2n-2}^{\ h_{2n-5}} \cdot h_{2n-5} = h_{2n-3} + h_{2n-2}. \]

Obtained equality holds by definition of the sequence \( \{h_n\}_{n \geq 1} \).

**Proposition 2.2.** The following formula is valid:

\[
h_{2n} = \begin{bmatrix}
q & \ h_3 & 1 \\
\ h_2 & 1 & \ h_5 \\
\ h_4 & 1 & \ h_7 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & \ h_{2n-3}^{\ h_{2n-5}} & 1 \\
0 & 0 & 0 & \cdots & 0 & \ h_{2n-1}^{\ h_{2n-4}} & 1
\end{bmatrix}.
\] (6)

**Proof.** Using (3), we have

\[ h_{2n} = 1 \cdot h_{2n-2} + \ h_{2n-1}^{\ h_{2n-4}} \cdot h_{2n-4} = h_{2n-2} + h_{2n-1}. \]
3 Main results

In this section we proved two recurrence formulas expressing the Horadam numbers $h_n$ by the determinant of tridiagonal matrix. As a consequence we received the corresponding formulas for the Fibonacci and Lucas numbers.

Proposition 3.1. The following formulas are valid:

\[
\begin{align*}
 h_{2n-1} &= \frac{1}{h_{1}h_{3}\cdots h_{2n-5}} \left| \begin{array}{ccccccc}
p & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-h_{2} & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -h_{4} & h_{1} & h_{1} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -h_{6} & h_{3} & h_{3} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -h_{2n-4} & h_{2n-7} & h_{2n-7} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -h_{2n-2} & h_{2n-5}
\end{array} \right|, \quad (7) \\

h_{2n} &= \frac{1}{h_{2}h_{4}\cdots h_{2n-4}} \left| \begin{array}{ccccccc}
p & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-h_{3} & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -h_{5} & h_{2} & h_{2} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -h_{7} & h_{4} & h_{4} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -h_{2n-3} & h_{2n-6} & h_{2n-6} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -h_{2n-1} & h_{2n-4}
\end{array} \right|. \quad (8)
\end{align*}
\]

Proof. We prove the formula (7). From (5) using (4), we have

\[
 h_{2n-1} = \frac{1}{h_{1}h_{3}\cdots h_{2n-5}} \left| \begin{array}{ccccccc}
p & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-h_{2} & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -h_{4} & h_{1} & h_{1} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -h_{6} & h_{3} & h_{3} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -h_{2n-4} & h_{2n-7} & h_{2n-7} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -h_{2n-2} & h_{2n-5}
\end{array} \right|.
\]

After obvious simple transformations, we get (7).

Formula (8) can be proved similarly. \(
\)

Example 3.2. Fibonacci numbers with odd indices:

\[
 F_{2n-1} = \frac{1}{F_{1}F_{3}\cdots F_{2n-5}} \left| \begin{array}{ccccccc}
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-F_{2} & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -F_{4} & F_{1} & F_{1} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -F_{6} & F_{3} & F_{3} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -F_{2n-4} & F_{2n-7} & F_{2n-7} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -F_{2n-2} & F_{2n-5}
\end{array} \right|.
\]

\]

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Example 3.3. The Fibonacci numbers with even indices:

$$F_{2n} = \frac{1}{F_2 F_4 \cdots F_{2n-4}} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -F_3 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -F_5 & F_2 & F_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -F_7 & F_4 & F_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -F_{2n-3} & F_{2n-6} & F_{2n-6} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -F_{2n-1} & F_{2n-4} \end{vmatrix}.$$  

Example 3.4. The Lucas numbers with odd indices:

$$L_{2n-1} = \frac{1}{L_1 L_3 \cdots L_{2n-5}} \begin{vmatrix} 2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -L_2 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -L_4 & L_1 & L_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -L_6 & L_3 & L_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -L_{2n-4} & L_{2n-7} & L_{2n-7} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -L_{2n-2} & L_{2n-5} \end{vmatrix}.$$  

Example 3.5. The Lucas numbers with even indices:

$$L_{2n} = \frac{1}{L_2 L_4 \cdots L_{2n-4}} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -L_3 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -L_5 & L_2 & L_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -L_7 & L_4 & L_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -L_{2n-3} & L_{2n-6} & L_{2n-6} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -L_{2n-1} & L_{2n-4} \end{vmatrix}.$$  

Note, that determinants of matrices, elements of which are classical or generalized Fibonacci numbers, in particular, studied in [1, 2, 3, 4, 6, 7, 8].

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References


