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The greatest common divisors of generalized Fibonacci and generalized Pell numbers

Boonyen Thongkam[1] and Nutcha Sailadda[2]

[1] Department of Mathematics, Faculty of Science
Ubon Ratchathani Rajabhat University
Ubon Ratchathani, 34000, Thailand
e-mail: boonyen.t@ubru.ac.th

[2] Department of Mathematics, Faculty of Education
Ubon Ratchathani Rajabhat University
Ubon Ratchathani, 34000, Thailand
e-mail: nutchasailadda@gmail.com

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Abstract: Abd-Elhameed and Zeyada have introduced the generalized sequence of numbers $(U_n^{a,b,r})_{n\geq 0}$ such that sequence generalizes both generalized Fibonacci numbers $(G_n^{a,b})_{n\geq 0}$ and generalized Pell numbers $(P_n^{a,b})_{n\geq 0}$. In the present paper, we show a study of the greatest common divisors of some $G_n^{a,b}$, $P_n^{a,b}$ and $U_n^{a,b,r}$.
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1 Introduction

Let $(F_n)_{n\geq 0}$ be the *Fibonacci sequence* given by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ and let $(L_n)_{n\geq 0}$ be the *Lucas sequence* given by $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$. Fibonacci numbers and Lucas numbers are famous for their special and amazing properties. An important identity of Fibonacci and Lucas numbers that they mutually have is $L_n = F_{n-1} + F_{n+1}$ for any $n \geq 1$. Let G_n be the *generalized Fibonacci numbers* given by $G_0^{a,b} = b - a$, $G_1^{a,b} = a$ and $G_{n+2}^{a,b} = G_{n+1}^{a,b} + G_n^{a,b}$ for $n \geq 0$. It is clear that such the generalized Fibonacci numbers are the generalization of both Fibonacci and Lucas numbers. In fact, we have $F_n = G_n^{1,1}$ and $L_n = G_n^{1,3}$. Koshy [5] has already proved that $G_{n+2}^{a,b} = aF_n + bF_{n+1}$.

Let $(P_n)_{n\geq 0}$ be the *Pell sequence* given by $P_0 = 0$, $P_1 = 1$, $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$ and let $(Q_n)_{n\geq 0}$ be the *Pell–Lucas sequence* given by $Q_0 = 2$, $Q_1 = 2$, $Q_{n+2} = 2Q_{n+1} + Q_n$ for$n \geq 0$. An important identity of Pell and Pell–Lucas numbers is $Q_n = P_{n-1} + P_{n+1}$ for any $n \geq 1$. Abd-Elhameed and Zeyada [1] have introduced the *generalized Pell numbers* as $P_0^{a,b} = b - 2a$, $P_1^{a,b} = a$ and $P_{n+2}^{a,b} = 2P_{n+1}^{a,b} + P_n^{a,b}$ for $n \geq 0$. It is clear that such the generalized Pell numbers are the generalization of both Pell and Pell–Lucas numbers. In fact, we have $P_n = P_n^{1,2}$ and $Q_n = P_n^{2,6}$. Furthermore, the authors [1] have introduced the new sequence of generalized numbers $(U_n^{a,b,r})_{n\geq 0}$ as

$$U_{n+2}^{a,b,r} = rU_{n+1}^{a,b,r} + U_n^{a,b,r} \text{ where } U_0^{a,b,r} = b - ra \text{ and } U_1^{a,b,r} = a.$$

We have $G_n^{a,b} = U_n^{a,b,1}$ and $P_n^{a,b} = U_n^{a,b,2}$. Thus, we have the two sequences $(G_n^{a,b})_{n\geq 0}$ and $(P_n^{a,b})_{n\geq 0}$ are particular sequences of the more general sequence $(U_n^{a,b,r})_{n\geq 0}$.

The greatest common divisor of positive integers a and b is the largest positive integer d such that a and b are both multiples of d. Let $\gcd(a, b)$ represent the greatest common divisor of a and b. The greatest common divisors of Fibonacci, Lucas, Pell and generalized Fibonacci numbers have been studied widely, which can be seen in [2, 3, 4, 5, 6]. It is well known that $\gcd(F_m, F_n) = F_{\gcd(m,n)}$, and $\gcd(P_m, P_n) = P_{\gcd(m,n)}$.

In this paper, we show our study of the greatest common divisors of $G_n^{a,b}$, $P_n^{a,b}$ and $U_n^{a,b,r}$.

2 The main results

The following theorem is well known and can be found many textbook on number theory.

Theorem 1. [7] *For every non-zero integers a and b, then $\gcd(a, b) = \gcd(a + bx, b)$ for any integer x.*

Theorem 2. [4, 5] *For every positive integers m, n, we have*

(1) *$n|m$ if and only if $F_n|F_m$,*

(2) *$n|m$ if and only if $P_n|P_m$.*

Theorem 3. [4, 5] *For every positive integers m, n, we have $\gcd(F_m, F_n) = F_{\gcd(m,n)}$, and $\gcd(P_m, P_n) = P_{\gcd(m,n)}$. This implies that $\gcd(F_n, F_{n+1}) = (P_n, P_{n+1}) = 1$.*

Theorem 4. *For every non-zero integers a, b, r and non-negative integer n, we have*

$$\gcd(U_n^{a,b,r}, U_{n+1}^{a,b,r}) = \gcd(a, b).$$

Proof. We will proceed by induction on n. It is clear in case $n = 0$. Now, assume that $\gcd(U_n^{a,b,r}, U_{n+1}^{a,b,r}) = \gcd(a, b)$ for non-negative integer n. Then we have

$$\begin{aligned}\gcd(U_{n+1}^{a,b,r}, U_{n+2}^{a,b,r}) &= \gcd(U_{n+1}^{a,b,r}, rU_{n+1}^{a,b,r} + U_n^{a,b,r}) \\ &= \gcd(U_{n+1}^{a,b,r}, U_n^{a,b,r}) \\ &= \gcd(a, b).\end{aligned}$$

This completes the proof of Theorem 4. □

Let $r = 1, 2$ in Theorem 4. We can get the corollary.

Corollary 5. *For every non-zero integers a, b and non-negative integer n, we have*

(1) $\gcd(G_n^{a,b}, G_{n+1}^{a,b}) = \gcd(a, b)$,

(2) $\gcd(P_n^{a,b}, P_{n+1}^{a,b}) = \gcd(a, b)$.

Corollary 6. *For non-negative integer n, we have*

(1) $\gcd(F_n, F_{n+1}) = 1$,

(2) $\gcd(L_n, L_{n+1}) = 1$,

(3) $\gcd(P_n, P_{n+1}) = 1$,

(4) $\gcd(Q_n, Q_{n+1}) = 2$.

Theorem 7. *For every non-zero integer a and non-negative integers n, m, we have*

$$\gcd(G_n^{a,a}, G_m^{a,a}) = |a| F_{\gcd(m,n)}.$$

Proof.

$$\begin{aligned}
\gcd(G_m^{a,a}, G_n^{a,a}) &= \gcd(aF_{m-2} + aF_{m-1}, aF_{n-2} + aF_{n-1}) \\
&= |a| \cdot \gcd(F_{m-2} + F_{m-1}, F_{n-2} + F_{n-1}) \\
&= |a| \cdot \gcd(F_m, F_n) \\
&= |a| F_{\gcd(m,n)}.
\end{aligned}$$

□

Lemma 8. *For every non-zero integers a, b, r and non-negative integer n, we have*

$$\gcd(G_n^{a,b}, G_n^{a,b+r}) = \gcd(aF_{n-2} + bF_{n-1}, rF_{n-1}).$$

Proof.

$$\begin{aligned}
\gcd(G_n^{a,b}, G_n^{a,b+r}) &= \gcd(aF_{n-2} + bF_{n-1}, aF_{n-2} + (b+r)F_{n-1}) \\
&= \gcd(aF_{n-2} + bF_{n-1}, aF_{n-2} + bF_{n-1} + rF_{n-1}) \\
&= \gcd(aF_{n-2} + bF_{n-1}, rF_{n-1}).
\end{aligned}$$

□

Corollary 9. *For every non-zero integers a, b, r and non-negative integer n, we have*

(1) $\gcd(G_n^{a,b}, G_n^{a,b+1}) = \gcd(a, F_{n-1})$,

(2) *if $r|b$, then* $\gcd(G_n^{a,b}, G_n^{a,b+r}) = \gcd(aF_{n-2}, rF_{n-1})$.

Theorem 10. *For every positive integer n, then $\gcd(F_n, L_n)$ is 1 or 2. Moreover, we have*

$$\gcd(F_n, L_n) = 2 \text{ if and only if } 3|n.$$

Proof. We set $a = b = 1$ and $r = 2$ in Lemma 8, then $\gcd(F_n, L_n) = \gcd(G_n^{1,1}, G_n^{1,3}) = \gcd(F_{n-2} + F_{n-1}, 2F_{n-1}) = \gcd(F_n, 2F_{n-1})$. Since $\gcd(F_n, F_{n+1}) = 1$, we have $\gcd(F_n, L_n)$ is 1 or 2. By Theorem 2, it is clear that $\gcd(F_n, L_n) = 2$ if and only if $3|n$. □

Table 1: $\gcd(F_n, L_n)$ for some values of n.

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
L_n	2	1	3	4	7	11	18	29	47	76	123
$\gcd(F_n, L_n)$	2	1	1	2	1	1	2	1	1	2	1

Theorem 11. *For every non-zero integers a, b, r and non-negative integer n, we have*

(1) $\gcd(U_{2n}^{a,b,r}, U_{2n+2}^{a,b,r}) = \gcd(ra, b)$,

(2) $\gcd(U_{2n+1)}^{a,b,r}, U_{2n+3}^{a,b,r}) = \gcd(a, rb)$.

Proof. We will proceed by induction on n. It is clear that $\gcd(U_0^{a,b,r}, U_2^{a,b,r}) = gcd(b - ra, b) = \gcd(ra, b)$ and $\gcd(U_1^{a,b,r}, U_3^{a,b,r}) = gcd(a, rb + a) = \gcd(a, rb)$.

First, we assume that $\gcd(U_{2n}^{a,b,r}, U_{2n+2}^{a,b,r}) = \gcd(ra, b)$. Then we have

$$\begin{aligned}
\gcd(U_{2n+2}^{a,b,r}, U_{2n+4}^{a,b,r}) &= \gcd(U_{2n+2}^{a,b,r}, rU_{2n+3}^{a,b,r} + U_{2n+2}^{a,b,r}) \\
&= \gcd(U_{2n+2}^{a,b,r}, rU_{2n+3}^{a,b,r}) \\
&= \gcd(U_{2n+2}^{a,b,r}, r(rU_{2n+2}^{a,b,r} + U_{2n+1}^{a,b,r})) \\
&= \gcd(U_{2n+2}^{a,b,r}, rU_{2n+1}^{a,b,r}) \\
&= \gcd(U_{2n+2}^{a,b,r}, U_{2n+2}^{a,b,r} - U_{2n}^{a,b,r}) \\
&= \gcd(U_{2n+2}^{a,b,r}, U_{2n}^{a,b,r}) \\
&= \gcd(ra, b).
\end{aligned}$$

Finally, we assume that $\gcd(U_{2n+1}^{a,b,r}, U_{2n+3}^{a,b,r}) = \gcd(a, rb)$. Then we have

$$\begin{aligned}
\gcd(U_{2n+3}^{a,b,r}, U_{2n+5}^{a,b,r}) &= \gcd(U_{2n+3}^{a,b,r}, rU_{2n+4}^{a,b,r} + U_{2n+3}^{a,b,r}) \\
&= \gcd(U_{2n+3}^{a,b,r}, rU_{2n+4}^{a,b,r}) \\
&= \gcd(U_{2n+3}^{a,b,r}, r(rU_{2n+3}^{a,b,r} + U_{2n+2}^{a,b,r})) \\
&= \gcd(U_{2n+3}^{a,b,r}, rU_{2n+2}^{a,b,r}) \\
&= \gcd(U_{2n+3}^{a,b,r}, U_{2n+3}^{a,b,r} - U_{2n+1}^{a,b,r}) \\
&= \gcd(U_{2n+3}^{a,b,r}, U_{2n+1}^{a,b,r}) \\
&= \gcd(a, rb).
\end{aligned}$$

This completes the proof of Theorem 11. □

Let $r = 1, 2$ in Theorem 11. We can get the corollary.

Corollary 12. *For every non-zero integers a, b and non-negative integer n, we have*

(1) $\gcd(G_n^{a,b}, G_{n+2}^{a,b}) = \gcd(a, b)$,

(2) $\gcd(P_n^{a,b}, P_{n+2}^{a,b}) = \begin{cases} \gcd(2a, b), & \text{if } n \text{ is even;} \\ \gcd(a, 2b), & \text{if } n \text{ is odd.} \end{cases}$

Corollary 13. *For non-negative integer n, we have*

(1) $\gcd(F_n, F_{n+2}) = 1$,

(2) $\gcd(L_n, L_{n+2}) = 1$,

(3) $\gcd(P_n, P_{n+2}) = \begin{cases} 2, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd,} \end{cases}$

(4) $\gcd(Q_n, Q_{n+2}) = 2$.

Theorem 14. *For every integers a, b and non-negative integer n, we have*

$$P_{n+2}^{a,b} = aP_n + bP_{n+1}.$$

Proof. We will proceed by induction. We will proceed by induction on n. We have $P_2^{a,b} = b = aP_0 + bP_1$ and $P_3^{a,b} = a + 2b = aP_1 + bP_2$. Now, we assume that $P_{k+2}^{a,b} = aP_k + bP_{k+1}$ for every $k \in \{0, 1, 2, \ldots, n\}$. Then

$$\begin{aligned} P_{n+3}^{a,b} &= 2P_{n+2}^{a,b} + P_{n+1}^{a,b} \\ &= 2(aP_n + bP_{n+1}) + (aP_{n-1} + bP_n) \\ &= a(2P_n + P_{n-1}) + b(2P_{n+1} + P_n) \\ &= aP_{n+1} + bP_{n+2}. \end{aligned}$$

This completes the proof of Theorem 14. □

Lemma 15. *For every non-zero integers a, b, r, s and non-negative integer n, we have*

$$\gcd(P_n^{a,b}, P_n^{a+r,b+s}) = \gcd(aP_{n-2} + bP_{n-1}, rP_{n-2} + sP_{n-1}).$$

Proof.

$$\begin{aligned} \gcd(P_n^{a,b}, P_n^{a+r,b+s}) &= \gcd(aP_{n-2} + bP_{n-1}, (a+r)P_{n-2} + (b+s)P_{n-1}) \\ &= \gcd(aP_{n-2} + bP_{n-1}, aP_{n-2} + bP_{n-1} + rP_{n-2} + sP_{n-1}) \\ &= \gcd(aP_{n-2} + bP_{n-1}, rP_{n-2} + sP_{n-1}). \end{aligned}$$ □

Theorem 16. *For every non-negative integer n, then $\gcd(P_n, Q_n)$ is 1 or 2. Moreover, we have $\gcd(P_n, Q_n) = 2$ if and only if n is even.*

Proof. We set $a = r = 1, b = 2$ and $s = 4$ in Lemma 15, then

$$\gcd(P_n, Q_n) = \gcd(P_n^{1,2}, P_n^{2,6}) = \gcd(P_{n-2} + 2P_{n-1}, P_{n-2} + 4P_{n-1}) = \gcd(P_n, 2P_{n-1}).$$

Since $\gcd(P_n, P_{n+1}) = 1$, we have $\gcd(P_n, Q_n)$ is 1 or 2. By Theorem 2, it is clear that $\gcd(P_n, Q_n) = 2$ if and only if n is even. □

Table 2: $\gcd(P_n, Q_n)$ for some values of n.

n	0	1	2	3	4	5	6	7	8	9	10
P_n	0	1	2	5	12	29	70	169	408	985	2,378
Q_n	2	2	6	14	34	82	198	478	1,154	2,786	6,726
$\gcd(P_n, Q_n)$	2	1	2	1	2	1	2	1	2	1	2

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