Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 24, 2018, No. 1, 94–96
DOI: 10.7546/nntdm.2018.24.1.94-96

Short note on a new arithmetic function

Krassimir T. Atanassov

Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria
e-mail: krat@bas.bg

Received: 4 July 2016 **Accepted:** 29 October 2017

Abstract: A new arithmetic function is introduced. It is illustrated with some examples with well known arithmetic functions, as, e.g., π- and φ-functions.
Keywords: Arithmetic functions φ, ψ and σ.
2010 Mathematics Subject Classification: 11A25.

First, let for every arithmetic function F such that for every natural number $n \geq 1$, $F(n) \leq n$, we define:

1. $F_0(n) = n$,
2. $F_1(n) = F(n)$,
3. for every $k \geq 1: \ F_{k+1}(n) = F(F_k(n))$.

Second, let us define the following new arithmetic function f_F, where F is the function, defined above, such that for every natural number $n \geq 2$:

$$f_F(n) = k \text{ if and only if for } k \geq 1, F_{k-1}(n) > 1 \text{ and } F_k(n) = 1. \tag{1}$$

For example, if F is the function π, determining the number of primes that are less than or equal to n (see, e.g. [3, 4]), then the values of the new function are given in the following table.

Table 1

n	$\pi(n)$	$f_\pi(n)$	n	$\pi(n)$	$f_\pi(n)$	n	$\pi(n)$	$f_\pi(n)$	n	$\pi(n)$	$f_\pi(n)$
1	1	1	9	4	3	126	30	5	5380	708	7
2	1	1	10	4	3	127	31	6	5381	709	8
3	2	2	11	5	4	128	31	6	5382	709	8
4	2	2	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	3	3	30	10	4	708	126	6	52710	5380	8
6	3	3	31	11	5	709	127	7	52711	5381	9
7	4	3	32	11	5	710	127	7	52712	5381	9
8	4	3	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

For the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $k, \alpha_1, \ldots, \alpha_k, k \geq 1$ are natural numbers and $p_1, \ldots, p_k$ are different primes, the following arithmetic functions are defined by:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1),\ \varphi(1) = 1,$$

(see, e.g. [3, 4]) and

$$\rho(n) = \prod_{i=1}^{k} (p_i^{\alpha_i} - p_i^{\alpha_i - 1} + \ldots + p_i^0(-1)^{\alpha_i}),\ \rho(1) = 1$$

(see [1, 2]).

Let

$$\Phi(n) = \max_{1 \leq k \leq n} \varphi(n).$$

Theorem 1. For every natural number $n \geq 2$

$$f_\pi(n) \leq f_\Phi(n). \tag{2}$$

Proof. For $n = 2$ the assertion is valid, because $f_\pi(2) = 1 = f_\Phi(2)$. Let us assume that Theorem 1 is valid for some natural number n. We will prove it for $n + 1$.

For number $n + 1$ there are two cases.

If $n + 1$ is a prime number, then

$$f_\pi(n + 1) = f_\pi(n) + 1$$

(by induction assumption)

$$\leq f_\Phi(n) + 1 = f_\Phi(n + 1).$$

If $n+1$ is not a prime number, then

$$f_\pi(n+1) = f_\pi(n)$$

(by (2) and induction assumption)

$$\leq f_\Phi(n) = f_\Phi(n+1). \qquad \square$$

Let

$$P(n) = \max_{1\leq k\leq n} \rho(n).$$

Theorem 2. For every natural number $n \geq 2$

$$f_P(n) = f_\Phi(n). \tag{3}$$

The proof of (3) is similar than the above one.

Function F can have more than one argument. For example, let for the two natural numbers $n \geq 1, s \geq 2$:

1. $F_0(n, s) = n,$
2. $F_1(n, s) = \left[\frac{n}{s}\right],$
3. for every $k \geq 1:\ F_{k+1}(n, s) = F(F_k(n, s)).$

Then for f_F the following theorem is valid.

Theorem 3. For every natural numbers $n \geq 1, s \geq 2$ and for the above defined function F:

$$f_F(n, s) \leq \log_s n.$$

Obviously, the equality exists for the case $n = s^k$ for some natural number k.

References

[1] Atanassov, K. (2001) Remarks on φ, σ, ψ and ρ functions. *Notes on Number Theory and Discrete Mathematics*, 7, 1, 1–3.

[2] Atanassov, K. (2009) A remark on an arithmetic function. Part 2. *Notes on Number Theory and Discrete Mathematics*, 15, 3, 21–22.

[3] Mitrinović, D., & Sándor, J. (1996) *Handbook of Number Theory*, Kluwer Academic Publishers.

[4] Nagell, T. (1950) *Introduction to Number Theory*, John Wiley & Sons, New York.