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Fourier series of sums of products of Bernoulli and Euler/Genocchi functions

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Abstract: We study the Fourier series of functions related to sum of products of Bernoulli polynomials and either Euler or Genocchi polynomials. As consequences, several new identities for the Bernoulli, Euler, and Genocchi functions and numbers are derived.

Keywords: Fourier series, Bernoulli polynomials, Euler polynomials, Genocchi polynomials.
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1 Introduction

We know that *Bernoulli, Euler and Genocchi numbers and polynomials* appear everywhere in mathematics (for example, see [1,6,14–16,18,21–23]). The *Bernoulli, Euler and Genocchi numbers* have been defined by the generating functions $\frac{t}{e^t-1} = \sum_{m\geq 0} B_m \frac{t^m}{m!}$, $\frac{2}{e^t+1} = \sum_{m\geq 0} E_m \frac{t^m}{m!}$, and $\frac{2t}{e^t+1} = \sum_{m\geq 0} G_m \frac{t^m}{m!}$, respectively. The *Bernoulli, Euler and Genocchi polynomials* $B_m(x)$, $E_m(x)$ and $G_m(x)$ have been given by the generating functions [1,4–6,12–19,21–23])

$$\frac{t}{e^t-1}e^{xt} = \sum_{m\geq 0} B_m(x)\frac{t^m}{m!},\ \frac{2}{e^t+1}e^{xt} = \sum_{m\geq 0} E_m(x)\frac{t^m}{m!},\ \frac{2t}{e^t+1}e^{xt} = \sum_{m\geq 0} G_m(x)\frac{t^m}{m!},$$

respectively, for any real number x, namely $x \in \mathbb{R}$. For instance, $B_0(x) = E_0(x) = G_1(x) = 1$, $G_0(x) = 0$, $B_1(x) = E_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$, $E_2(x) = x^2 - x$, $G_2(x) = 2x - 1$, $B_3(x) = x^3 - 3x^2/2 + x/2$, and $E_3(x) = x^3 - 3x^2/2 + 1/4$, and $G_3(x) = 3x^2 - 3x$. Clearly, $G_m(x) = mE_{m-1}(x)$, $\frac{d}{dx}G_m(x) = mG_{m-1}(x)$ and $G_m(x+1) + G_m(x) = 2mx^{m-1}$, for all $m \geq 1$. For $u \in \mathbb{R}$, we denote the fractional part of u by $\langle u\rangle = u - \lfloor u\rfloor \in [0,1)$. In this paper, we are interested in six functions related to Bernoulli polynomials: $\tilde{\alpha}_m(x) = \alpha_m(\langle x\rangle)$, $\tilde{\beta}_m(x) = \beta_m(\langle x\rangle)$, $\tilde{\gamma}_m(x) = \gamma_m(\langle x\rangle)$, $\tilde{\delta}_m(x) = \delta_m(\langle x\rangle)$, $\tilde{\eta}_m(x) = \eta_m(\langle x\rangle)$ and $\tilde{\theta}_m(x) = \theta_m(\langle x\rangle)$, where

$$\begin{aligned}
\alpha_m(x) &= \sum_{k=0}^{m} B_k(x)E_{m-k}(x), & \beta_m(x) &= \sum_{k=0}^{m-1} B_k(x)G_{m-k}(x),\\
\gamma_m(x) &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}B_k(x)E_{m-k}(x), & \delta_m(x) &= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!}B_k(x)G_{m-k}(x),\\
\eta_m(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}B_k(x)E_{m-k}(x), & \theta_m(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}B_k(x)G_{m-k}(x),
\end{aligned}$$

where $m \geq 1$, for $\alpha_m(x), \gamma_m(x)$, and $m \geq 2$, for $\beta_m(x), \delta_m(x), \eta_m(x), \theta_m(x)$. We recall the following facts about Bernoulli functions:

$$-m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m} = \tilde{B}_m(x),\ m \geq 2, \tag{1}$$

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z},\\ 0, & x \in \mathbb{Z}. \end{cases} \tag{2}$$

where $\tilde{B}_m(x) = B_m(\langle x\rangle)$.

The Fourier series of a periodic function $f(x)$ with period 1 is given by $\sum_{n=-\infty}^{\infty} f_n e^{2\pi inx}$, where the coefficients f_n are given by $f_n = \int_0^1 f(x)e^{-2\pi inx}dx$ (for example, see [2,3,8–11,16,20,24–26]), where $i^2 = -1$.

The aim of this paper is to consider the *Fourier series* of $\tilde{\alpha}_m(x)$, $\tilde{\beta}_m(x)$, $\tilde{\gamma}_m(x)$, $\tilde{\delta}_m(x)$, $\tilde{\eta}_m(x)$ and $\tilde{\theta}_m(x)$, which lead to several new identities for the Bernoulli functions and numbers, see the next three sections.

2 The functions $\tilde{\alpha}_m$ and $\tilde{\beta}_m$

In this section, we consider the functions $\tilde{\alpha}_m, (m \geq 1)$, and $\tilde{\beta}_m, (m \geq 2)$ on $\mathbb{R}$, each of which is periodic with period 1. The Fourier series of $\tilde{\alpha}_m$ and $\tilde{\beta}_m$ are $\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}$ and $\sum_{n=-\infty}^{\infty} \tilde{A}_n^{(m)} e^{2\pi i n x}$, where $A_n^{(m)} = \int_0^1 \tilde{\alpha}_m(x) e^{-2\pi i n x} dx = \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx$ and $\tilde{A}_n^{(m)} = \int_0^1 \tilde{\beta}_m(x) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx$, respectively.

Define $\Lambda_m = 2B_m - E_{m-1} - 2\sum_{k=0}^{m} B_k E_{m-k} + 2\delta_{m,1}$, for $m \geq 1$, and $\tilde{\Lambda}_m = -G_{m-1} - 2\sum_{k=0}^{m-2} B_k G_{m-k} + 2\delta_{m,2}$, for $m \geq 2$. To proceed further, we note the following lemma.

Lemma 1. *For* $m \geq 1$,

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \Lambda_{m+1},$$
$$\int_0^1 \beta_m(x) dx = \frac{1}{m+2} \tilde{\Lambda}_{m+1}.$$

Moreover, $\alpha_m(1) = \alpha_m(0)$ *if and only if* $\Lambda_m = 0$*, and* $\beta_m(1) = \beta_m(0)$ *if and only if* $\tilde{\Lambda}_m = 0$.

Proof. Recall that $\frac{d}{dx} B_k(x) = k B_{k-1}(x)$ and $\frac{d}{dx} E_k(x) = k E_{k-1}(x)$, for all $k \geq 1$. So, by the definitions, we have

$$\begin{aligned}\frac{d}{dx}\alpha_m(x)dx &= \sum_{k=0}^{m} k B_{k-1}(x) E_{m-k}(x) + \sum_{k=0}^{m} (m-k) B_k(x) E_{m-1-k}(x) \\ &= \sum_{k=0}^{m-1} (k+1) B_k(x) E_{m-1-k}(x) + \sum_{k=0}^{m-1} (m-k) B_k(x) E_{m-1-k}(x) \\ &= (m+1)\alpha_{m-1}(x).\end{aligned}$$

Thus, $\alpha_m(x) = \frac{1}{m+2} \frac{d}{dx} \alpha_{m+1}(x)$, which implies

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)) = \frac{1}{m+2} \Lambda_{m+1},$$

as claimed.

Recall that $\frac{d}{dx} G_k(x) = k G_{k-1}(x)$, for all $k \geq 1$. So, by the definitions,

$$\begin{aligned}\frac{d}{dx}\beta_m(x) &= \sum_{k=0}^{m} k B_{k-1}(x) G_{m-k}(x) + \sum_{k=0}^{m} (m-k) B_k(x) G_{m-1-k}(x) \\ &= \sum_{k=0}^{m-1} (k+1) B_k(x) G_{m-1-k}(x) + \sum_{k=0}^{m-1} (m-k) B_k(x) G_{m-1-k}(x) \\ &= (m+1)\beta_{m-1}(x).\end{aligned}$$

Thus, $\beta_m(x) = \frac{1}{m+2} \frac{d}{dx} \beta_{m+1}(x)$, which implies

$$\int_0^1 \beta_m(x) dx = \frac{1}{m+2}(\beta_{m+1}(1) - \beta_{m+1}(0)) = \frac{1}{m+2} \tilde{\Lambda}_{m+1},$$

as claimed. □

Now, we are ready to determine the Fourier coefficients $A_n^{(m)}$ and $\tilde{A}_n^{(m)}$. The case $n = 0$ follows from Lemma 1, that is,

$$A_0^{(m)} = \frac{1}{m+2}\Lambda_{m+1}, \quad \tilde{A}_0^{(m)} = \frac{1}{m+2}\tilde{\Lambda}_{m+1}. \tag{3}$$

Thus, let us assume that $n \neq 0$. By Lemma 1, we have

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx = \frac{1}{2\pi i n}\int_0^1 \frac{d}{dx}\alpha_m(x) e^{-2\pi i n x} dx - \frac{1}{2\pi i n}\, \alpha_m(x) e^{-2\pi i n x}\Big|_{x=0}^{x=1} \\ &= \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx - \frac{1}{2\pi i n}(\alpha_m(1) - \alpha_m(0)) \\ &= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{\Lambda_m}{2\pi i n}. \end{aligned}$$

Note that $A_n^{(1)} = \int_0^1 (2x-1) e^{-2\pi i n x} dx = -\frac{2}{2\pi i n}$ and $\tilde{A}_n^{(2)} = \int_0^1 (3x - 3/2) e^{-2\pi i n x} dx = -\frac{3}{2\pi i n}$. So, by induction on m, we obtain

$$A_n^{(m)} = -\frac{(m+1)!}{(2\pi i n)^m} - \frac{1}{m+2}\sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j}\Lambda_{m+1-j}, \tag{4}$$

where $m \geq 1$ and $(x)_j = x(x-1)\cdots(x-j+1)$ with $(x)_0 = 1$. Similarly, one can show that

$$\tilde{A}_n^{(m)} = -\frac{(m+1)_{m-1}}{(2\pi i n)^{m-1}} - \sum_{j=1}^{m-2} \frac{(m+1)_{j-1}}{(2\pi i n)^j}\tilde{\Lambda}_{m+1-j}, \tag{5}$$

for $m \geq 2$.

Note that the functions $\tilde{\alpha}_m$ and $\tilde{\beta}_m(x)$ are piecewise C^∞. Moreover, the functions $\tilde{\alpha}_m$ and $\tilde{\beta}_m$ are continuous for those integers m with $\Lambda_m = 0, (m \geq 1)$, and $\tilde{\Lambda}_m = 0, (m \geq 2)$, respectively, and discontinuous with jump discontinuities at integers for those integers m with $\Lambda_m \neq 0, (m \geq 1)$, and $\tilde{\Lambda}_m \neq 0, (m \geq 2)$, respectively.

2.1 Case $\Lambda_m = 0$ ($\tilde{\Lambda}_m = 0$)

Assume first that m is an integer with $\Lambda_m = 0, (m \geq 1)$ ($\tilde{\Lambda}_m = 0, (m \geq 2)$). Then $\alpha_m(1) = \alpha_m(0)$ ($\beta_m(1) = \beta_m(0)$). So, the functions $\tilde{\alpha}_m$ and $\tilde{\beta}_m$ are piecewise C^∞ and continuous. Thus, the Fourier series of $\tilde{\alpha}_m$ and $\tilde{\beta}_m$ converge uniformly to $\tilde{\alpha}_m$ and $\tilde{\beta}_m$, respectively. So, by (3), (4) and (5), we have

$$\begin{aligned} \tilde{\alpha}_m(x) &= \frac{\Lambda_{m+1}}{m+2} + \sum_{n\in\mathbb{Z}'}\left\{-\frac{(m+1)!}{(2\pi i n)^m} - \frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi i n)^j}\Lambda_{m+1-j}\right\} e^{2\pi i n x} \\ &= \frac{\Lambda_{m+1}}{m+2} - \frac{1}{m+2}\sum_{j=1}^{m-1}\binom{m+2}{j}\Lambda_{m+1-j}\, j!\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi i n x}}{(2\pi i n)^j} - (m+1)!\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi i n x}}{(2\pi i n)^m}, \\ \tilde{\beta}_m(x) &= \frac{\tilde{\Lambda}_{m+1}}{m+2} + \sum_{n\in\mathbb{Z}'}\left\{-\frac{(m+1)_{m-1}}{(2\pi i n)^{m-1}} - \frac{1}{m+2}\sum_{j=1}^{m-2}\frac{(m+2)_j}{(2\pi i n)^j}\tilde{\Lambda}_{m+1-j}\right\} e^{2\pi i n x} \\ &= \frac{\tilde{\Lambda}_{m+1}}{m+2} - \frac{1}{m+2}\sum_{j=1}^{m-2}\binom{m+2}{j}\tilde{\Lambda}_{m+1-j}\, j!\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi i n x}}{(2\pi i n)^j} - \frac{(m+1)!}{2}\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi i n x}}{(2\pi i n)^{m-1}}, \end{aligned}$$

where we define $\mathbb{Z}' = \mathbb{Z}\backslash\{0\}$. Thus, by (1) and (2), we obtain

$$\tilde{\alpha}_m(x) = \frac{\Lambda_{m+1}}{m+2} + \frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Lambda_{m+1-j}\tilde{B}_j(x) + (m+1)\tilde{B}_m(x)$$
$$+ \Lambda_m \begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}, \end{cases}$$
$$\tilde{\beta}_m(x) = \frac{\tilde{\Lambda}_{m+1}}{m+2} + \frac{1}{m+2}\sum_{j=2}^{m-2}\binom{m+2}{j}\tilde{\Lambda}_{m+1-j}\tilde{B}_j(x) + \binom{m+1}{2}\tilde{B}_{m-1}(x)$$
$$+ \tilde{\Lambda}_m \begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

for all $x \in \mathbb{R}$. Thus, we can state the following results.

Theorem 2. *Let m be a positive integer with $\Lambda_m = 0$. Then the function $\tilde{\alpha}_m(x)$ has the Fourier series expansion*

$$\tilde{\alpha}_m(x) = \frac{\Lambda_{m+1}}{m+2} + \sum_{n\in\mathbb{Z}'}\left\{-\frac{(m+1)!}{(2\pi in)^m} - \frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Lambda_{m+1-j}\right\}e^{2\pi inx},$$

for all $x \in \mathbb{R}$, where the convergence is uniform. Moreover,

$$\tilde{\alpha}_m(x) = \frac{\Lambda_{m+1}}{m+2} + \frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Lambda_{m+1-j}\tilde{B}_j(x) + (m+1)\tilde{B}_m(x).$$

Theorem 3. *Let m be an integer ≥ 2, with $\tilde{\Lambda}_m = 0$. Then the function $\tilde{\beta}_m(x)$ has the Fourier series expansion*

$$\tilde{\beta}_m(x) = \frac{\tilde{\Lambda}_{m+1}}{m+2} + \sum_{n\in\mathbb{Z}'}\left\{-\frac{(m+1)_{m-1}}{(2\pi in)^{m-1}} - \frac{1}{m+2}\sum_{j=1}^{m-2}\frac{(m+2)_j}{(2\pi in)^j}\tilde{\Lambda}_{m+1-j}\right\}e^{2\pi inx},$$

for all $x \in \mathbb{R}$, where the convergence is uniform. Moreover,

$$\tilde{\beta}_m(x) = \frac{\tilde{\Lambda}_{m+1}}{m+2} + \frac{1}{m+2}\sum_{j=2}^{m-2}\binom{m+2}{j}\tilde{\Lambda}_{m+1-j}\tilde{B}_j(x) + \binom{m+1}{2}\tilde{B}_{m-1}(x).$$

2.2 Case $\Lambda_m \neq 0$ ($\tilde{\Lambda}_m \neq 0$)

Assume next that m is an integer with $\Lambda_m \neq 0, (m \geq 1)$ ($\tilde{\Lambda}_m \neq 0, (m \geq 2)$). Then $\alpha_m(1) \neq \alpha_m(0)$ ($\beta(1) \neq \beta(0)$). So, the functions $\tilde{\alpha}_m$ and $\tilde{\beta}_m$ are pointwise C^∞ and discontinuous with jump discontinuities at integers. Thus, the Fourier series of $\tilde{\alpha}_m(x)$ and $\tilde{\beta}_m$ converge pointwise to $\tilde{\alpha}_m(x)$ and $\tilde{\beta}_m$ for all $x \notin \mathbb{Z}$, and converge to

$$\frac{\alpha_m(1) + \alpha_m(0)}{2} = B_m - \frac{1}{2}E_{m-1},$$
$$\frac{\beta_m(1) + \beta_m(0)}{2} = B_{m-1} - \frac{1}{2}G_{m-1},$$

for all $x \in \mathbb{Z}$. Then, by Theorems 2 and 3, we obtain the following results.

Theorem 4. *Let m be a positive integer with $\Lambda_m \neq 0$. Then*

$$\frac{\Lambda_{m+1}}{m+2} + \sum_{n\in\mathbb{Z}'} \left\{ -\frac{(m+1)!}{(2\pi in)^m} - \frac{1}{m+2}\sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi in)^j}\Lambda_{m+1-j} \right\} e^{2\pi inx}$$

equals $\tilde{\alpha}_m(x)$ for all $x \notin \mathbb{Z}$ and $B_m - \frac{1}{2}E_{m-1}$ for all $x \in \mathbb{Z}$, where the convergence is pointwise. Moreover,

$$\frac{\Lambda_{m+1}}{m+2} + \frac{1}{m+2}\sum_{j=1}^{m-1}\binom{m+2}{j}\Lambda_{m+1-j}\tilde{B}_j(x) + (m+1)\tilde{B}_m(x)$$

equals $\tilde{\alpha}_m(x)$ for all $x \notin \mathbb{Z}$ and

$$\frac{\Lambda_{m+1}}{m+2} + \frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Lambda_{m+1-j}\tilde{B}_j(x) + (m+1)\tilde{B}_m(x)$$

equals $B_m - \frac{1}{2}E_{m-1}$ for all $x \in \mathbb{Z}$.

Theorem 5. *Let m be an integer ≥ 2 with $\tilde{\Lambda}_m \neq 0$. Then*

$$\frac{\tilde{\Lambda}_{m+1}}{m+2} + \sum_{n\in\mathbb{Z}'} \left\{ -\frac{(m+1)_{m-1}}{(2\pi in)^{m-1}} - \frac{1}{m+2}\sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi in)^j}\tilde{\Lambda}_{m+1-j} \right\} e^{2\pi inx}$$

equals $\tilde{\beta}_m(x)$ for all $x \notin \mathbb{Z}$ and $B_{m-1} - \frac{1}{2}G_{m-1}$ for all $x \in \mathbb{Z}$, where the convergence is pointwise. Moreover,

$$\frac{\tilde{\Lambda}_{m+1}}{m+2} + \frac{1}{m+2}\sum_{j=1}^{m-2}\binom{m+2}{j}\tilde{\Lambda}_{m+1-j}\tilde{B}_j(x) + \binom{m+1}{2}\tilde{B}_{m-1}(x)$$

equals $\tilde{\beta}_m(x)$ for all $x \notin \mathbb{Z}$ and

$$\frac{\tilde{\Lambda}_{m+1}}{m+2} + \frac{1}{m+2}\sum_{j=2}^{m-2}\binom{m+2}{j}\tilde{\Lambda}_{m+1-j}\tilde{B}_j(x) + \binom{m+1}{2}\tilde{B}_{m-1}(x)$$

equals $B_{m-1} - \frac{1}{2}G_{m-1}$ for all $x \in \mathbb{Z}$.

In [13, 19, 20], it has been shown that

$$\int_0^1 \alpha_m(x)dx = -\frac{2E_{m+1}}{m+1} + \frac{2}{m+1}\sum_{k=1}^{m-1}\sum_{\ell=k+1}^{m}(-1)^{k+\ell}\frac{\binom{m+1}{\ell}}{\binom{m}{k}}B_\ell E_{m+1-\ell}.$$

Thus, by Lemma 1, we establish the following identity

$$\int_0^1 \alpha_m(x)dx = -\frac{2E_{m+1}}{m+1} + \frac{2}{m+1}\sum_{k=1}^{m-1}\sum_{\ell=k+1}^{m}(-1)^{k+\ell}\frac{\binom{m+1}{\ell}}{\binom{m}{k}}B_\ell E_{m+1-\ell} = \frac{\Lambda_{m+1}}{m+2}.$$

Theorems 2, 3, 4 and 5 suggest the following question: For what values of integers $m \geq 1$ does $\Lambda_m = 0$ ($\tilde{\Lambda}_m = 0$) hold?

3 The functions $\tilde{\gamma}_m$ and $\tilde{\delta}_m$

In this section, we consider the functions $\tilde{\gamma}_m, (m \geq 1)$, and $\tilde{\delta}_m, (m \geq 2)$ on $\mathbb{R}$, each of which is periodic with period 1.

The Fourier series of $\tilde{\gamma}_m$ and $\tilde{\delta}_m$ are $\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi inx}$ and $\sum_{n=-\infty}^{\infty} \tilde{B}_n^{(m)} e^{2\pi inx}$, where $B_n^{(m)} = \int_0^1 \tilde{\gamma}_m(x) e^{-2\pi inx} dx = \int_0^1 \gamma_m(x) e^{-2\pi inx} dx$ and $\tilde{B}_n^{(m)} = \int_0^1 \tilde{\delta}_m(x) e^{-2\pi inx} dx = \int_0^1 \delta_m(x) e^{-2\pi inx} dx$, respectively.

Define $\Omega_m = \frac{2B_m}{m!} - \frac{E_{m-1}}{(m-1)!} - 2\sum_{k=0}^{m} \frac{B_k E_{m-k}}{k!(m-k)!} + \frac{2\delta_{m,1}}{(m-1)!}$, for $m \geq 1$, and $\tilde{\Omega}_m = -2\sum_{k=0}^{m-2} \frac{B_k G_{m-k}}{k!(m-k)!} - \frac{G_{m-1}}{(m-1)!} + \frac{2\delta_{m,2}}{(m-1)!}$, for $m \geq 2$.

To proceed further, we note the following lemma.

Lemma 6. *For* $m \geq 1$,

$$\int_0^1 \gamma_m(x) dx = \frac{1}{2}\Omega_{m+1},$$
$$\int_0^1 \delta_m(x) dx = \frac{1}{2}\tilde{\Omega}_{m+1},$$

Moreover, $\gamma_m(1) = \gamma_m(0)$ *if and only if* $\Omega_m = 0$, *and* $\delta_m(1) = \delta_m(0)$ *if and only if* $\tilde{\Omega}_m = 0$.

Proof. Recall that $\frac{d}{dx}B_k(x) = kB_{k-1}(x)$ and $\frac{d}{dx}E_k(x) = kE_{k-1}(x)$, for all $k \geq 1$. So, by the definitions, we have

$$\begin{aligned}\frac{d}{dx}\gamma_m(x) &= \sum_{k=0}^{m} \frac{kB_{k-1}(x)E_{m-k}(x)}{k!(m-k)!} + \sum_{k=0}^{m} \frac{(m-k)B_k(x)E_{m-1-k}(x)}{k!(m-k)!} \\ &= \sum_{k=0}^{m-1} \frac{B_k(x)E_{m-1-k}(x)}{k!(m-1-k)!} + \sum_{k=0}^{m-1} \frac{B_k(x)E_{m-1-k}(x)}{k!(m-1-k)!} = 2\gamma_{m-1}(x).\end{aligned}$$

Thus, $\gamma_m(x) = \frac{1}{2}\frac{d}{dx}\gamma_{m+1}(x)$, which gives $\int_0^1 \gamma_m(x) = \frac{1}{2}(\gamma_{m+1}(1) - \gamma_{m+1}(0)) = \frac{1}{2}\Omega_{m+1}$, as claimed.

Recall that $\frac{d}{dx}G_k(x) = kG_{k-1}(x)$, for all $k \geq 1$. So, by the definitions,

$$\begin{aligned}\frac{d}{dx}\delta_m(x) &= \sum_{k=0}^{m-1} \frac{kB_{k-1}(x)G_{m-k}(x)}{k!(m-k)!} + \sum_{k=0}^{m-1} \frac{(m-k)B_k(x)G_{m-1-k}(x)}{k!(m-k)!} \\ &= \sum_{k=0}^{m-2} \frac{B_k(x)G_{m-1-k}(x)}{k!(m-1-k)!} + \sum_{k=0}^{m-2} \frac{B_k(x)G_{m-1-k}(x)}{k!(m-1-k)!} = 2\delta_{m-1}(x).\end{aligned}$$

Thus, $\delta_m(x) = \frac{1}{2}\frac{d}{dx}\delta_{m+1}(x)$, which implies $\int_0^1 \delta_m(x) = \frac{1}{2}(\delta_{m+1}(1) - \delta_{m+1}(0)) = \frac{1}{2}\tilde{\Omega}_{m+1}$, as claimed. □

Now, we are ready to determine the Fourier coefficients $B_n^{(m)}$, and $\tilde{B}_n^{(m)}$. The case $n = 0$ follows from Lemma 6, that is,

$$B_0^{(m)} = \frac{1}{2}\Omega_{m+1}, \quad \tilde{B}_0^{(m)} = \frac{1}{m+2}\tilde{\Omega}_{m+1}. \tag{6}$$

Thus, let us assume that $n \neq 0$. By Lemma 6, we have

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx = \frac{1}{2\pi in}\int_0^1 \frac{d}{dx}\gamma_m(x)e^{-2\pi inx}dx - \frac{1}{2\pi in}\,\gamma_m(x)e^{-2\pi inx}\Big|_{x=0}^{x=1} \\ &= \frac{2}{2\pi in}\int_0^1 \gamma_{m-1}(x)e^{-2\pi inx}dx - \frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) \\ &= \frac{2}{2\pi in}B_n^{(m-1)} - \frac{\Omega_m}{2\pi in}. \end{aligned}$$

Similarly,

$$\tilde{B}_n^{(m)} = \frac{2}{2\pi in}\tilde{B}_n^{(m-1)} - \frac{\tilde{\Omega}_m}{2\pi in}.$$

Note that $B_n^{(1)} = \int_0^1 (2x-1)e^{-2\pi inx}dx = -\frac{2}{2\pi in}$ and $\tilde{B}_n^{(2)} = \int_0^1 (2x-1)e^{-2\pi inx}dx = -\frac{2}{2\pi in}$. So, by induction on m, we obtain

$$B_n^{(m)} = -\frac{1}{(\pi in)^m} - \sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m+1-j}, \tag{7}$$

where $m \geq 1$ and $(x)_j = x(x-1)\cdots(x-j+1)$ with $(x)_0 = 1$. Similarly, one can show that

$$\tilde{B}_n^{(m)} = -\frac{1}{(\pi in)^{m-1}} - \sum_{j=1}^{m-2} \frac{2^{j-1}}{(2\pi in)^j}\tilde{\Omega}_{m+1-j}, \tag{8}$$

for $m \geq 2$.

Note that the functions $\tilde{\gamma}_m$, and $\tilde{\delta}_m$ are piecewise C^∞. Moreover, the functions $\tilde{\gamma}_m$ and $\tilde{\delta}_m$ are continuous for those integers m with $\Omega_m = 0, (m \geq 1)$, and $\tilde{\Omega}_m = 0, (m \geq 2)$, respectively, and discontinuous with jump discontinuities at integers for those integers m with $\Omega_m \neq 0, (m \geq 1)$, and $\tilde{\Omega}_m \neq 0, (m \geq 2)$, respectively.

3.1 Case $\Omega_m = 0$ ($\tilde{\Omega}_m = 0$)

Assume first that m is a positive integer with $\Omega_m = 0$ ($\tilde{\Omega}_m = 0$). Then $\gamma_m(1) = \gamma_m(0)$ ($\delta_m(1) = \delta_m(0)$). So, the functions $\tilde{\gamma}_m$ and $\tilde{\delta}_m$ are piecewise C^∞ and continuous. Thus, the Fourier series of $\tilde{\gamma}_m$ and $\tilde{\delta}_m$ converge uniformly to $\tilde{\gamma}_m$ and $\tilde{\delta}_m$, respectively. So, by (6), (7) and (8), we have

$$\begin{aligned} \tilde{\gamma}_m(x) &= \frac{\Omega_{m+1}}{2} - \sum_{n\in\mathbb{Z}'}\left\{\frac{1}{(\pi in)^m} + \sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m+1-j}\right\}e^{2\pi inx} \\ &= \frac{\Omega_{m+1}}{2} + \sum_{j=1}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m+1-j}(-j!)\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{(2\pi in)^j} - 2^m\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{(2\pi in)^m}, \\ \tilde{\delta}_m(x) &= \frac{\tilde{\Omega}_{m+1}}{2} + \sum_{n\in\mathbb{Z}'}\left\{-\frac{2^{m-1}}{(2\pi in)^{m-1}} - \sum_{j=1}^{m-2}\frac{2^{j-1}}{(2\pi in)^j}\tilde{\Omega}_{m+1-j}\right\}e^{2\pi inx} \\ &= \frac{\tilde{\Omega}_{m+1}}{2} - \sum_{j=1}^{m-2}\frac{2^{j-1}}{j!}\tilde{\Omega}_{m+1-j}j!\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{(2\pi in)^j} - 2^{m-1}\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{(2\pi in)^{m-1}}. \end{aligned}$$

Thus, by (1) and (2), we obtain

$$\tilde{\gamma}_m(x) = \frac{\Omega_{m+1}}{2} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!}\Omega_{m+1-j}\tilde{B}_j(x) + \frac{2^m}{m!}\tilde{B}_m(x) + \Omega_m \begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}, \end{cases}$$

$$\tilde{\delta}_m(x) = \frac{\tilde{\Omega}_{m+1}}{2} + \sum_{j=2}^{m-2} \frac{2^{j-1}}{j!}\tilde{\Omega}_{m+1-j}\tilde{B}_j(x) + \frac{2^{m-1}}{(m-1)!}\tilde{B}_{m-1}(x) + \tilde{\Omega}_m \begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

for all $x \in \mathbb{R}$. Thus, we can state the following results.

Theorem 7. *Let m be a positive integer with $\Omega_m = 0$. Then the function $\tilde{\gamma}_m(x)$ has the Fourier series expansion*

$$\tilde{\gamma}_m(x) = \frac{\Omega_{m+1}}{2} + \sum_{n \in \mathbb{Z}'} \left\{ -\frac{1}{(\pi i n)^m} - \sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m+1-j} \right\} e^{2\pi i n x},$$

for all $x \in \mathbb{R}$, where the convergence is uniform. Moreover,

$$\tilde{\gamma}_m(x) = \sum_{j=0, j\neq 1}^{m-1} \frac{2^{j-1}}{j!}\Omega_{m+1-j}\tilde{B}_j(x) + \frac{2^m}{m!}\tilde{B}_m(x),$$

for all $x \in \mathbb{R}$.

Theorem 8. *Let m be an integer ≥ 2 with $\tilde{\Omega}_m = 0$. Then the function $\tilde{\delta}_m(x)$ has the Fourier series expansion*

$$\tilde{\delta}_m(x) = \frac{\tilde{\Omega}_{m+1}}{2} + \sum_{n \in \mathbb{Z}'} \left\{ -\frac{1}{(\pi i n)^{m-1}} - \sum_{j=1}^{m-2} \frac{2^{j-1}}{(2\pi i n)^j}\tilde{\Omega}_{m+1-j} \right\} e^{2\pi i n x},$$

for all $x \in \mathbb{R}$, where the convergence is uniform. Moreover,

$$\tilde{\delta}_m(x) = \frac{\tilde{\Omega}_{m+1}}{2} + \sum_{j=2}^{m-2} \frac{2^{j-1}}{j!}\tilde{\Omega}_{m+1-j}\tilde{B}_j(x) + \frac{2^{m-1}}{(m-1)!}\tilde{B}_{m-1}(x),$$

for all $x \in \mathbb{R}$.

3.2 Case $\Omega_m \neq 0$ ($\tilde{\Omega}_m \neq 0$)

Assume next that m is a positive integer with $\Omega_m \neq 0, (m \geq 1)$ ($\tilde{\Omega}_m \neq 0, (m \geq 2)$). Then $\gamma_m(1) \neq \gamma_m(0)$ ($\delta(1) \neq \delta(0)$). So, the functions $\tilde{\gamma}_m$ and $\tilde{\delta}_m$ are piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus, the Fourier series of $\tilde{\gamma}_m(x)$ and $\tilde{\delta}_m$ converge pointwise to $\tilde{\gamma}_m(x)$ and $\tilde{\delta}_m$ for all $x \notin \mathbb{Z}$, and converge to

$$\frac{\gamma_m(1) + \gamma_m(0)}{2} = \frac{B_m}{m!} - \frac{E_{m-1}}{2(m-1)!} = \frac{2^m}{m!}B_m,$$

$$\frac{\delta_m(1) + \delta_m(0)}{2} = \frac{B_{m-1}}{(m-1)!} - \frac{G_{m-1}}{2(m-1)!} = \frac{2^{m-1}}{(m-1)!}B_{m-1},$$

for all $x \in \mathbb{Z}$. Then, by Theorems 7 and 8, we obtain the following results.

Theorem 9. *Let m be a positive integer with $\Omega_m \neq 0$. Then*

$$\frac{\Omega_{m+1}}{2} - \sum_{n\in\mathbb{Z}'} \left\{ \frac{1}{(\pi i n)^m} + \sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m+1-j} \right\} e^{2\pi i n x}$$

equals $\tilde{\gamma}_m(x)$ for all $x \notin \mathbb{Z}$ and $\frac{2^m}{m!} B_m$ for all $x \in \mathbb{Z}$, where the convergence is pointwise. Moreover,

$$\sum_{j=0}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m+1-j} \tilde{B}_j(x) + \frac{2^m}{m!} \tilde{B}_m(x)$$

equals $\tilde{\gamma}_m(x)$ for all $x \notin \mathbb{Z}$ and

$$\sum_{j=0, j\neq 1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m+1-j} \tilde{B}_j(x) + \frac{2^m}{m!} \tilde{B}_m(x)$$

equals $\frac{2^{m-1}}{(m-1)!} B_{m-1}$ for all $x \in \mathbb{Z}$.

Theorem 10. *Let m be an integer $m \geq 2$ with $\tilde{\Omega}_m \neq 0$. Then*

$$\frac{\tilde{\Omega}_{m+1}}{2} - \sum_{n\in\mathbb{Z}'} \left\{ \frac{1}{(\pi i n)^{m-1}} + \sum_{j=1}^{m-2} \frac{2^{j-1}}{(2\pi i n)^j} \tilde{\Omega}_{m+1-j} \right\} e^{2\pi i n x}$$

equals $\tilde{\delta}_m(x)$ for all $x \notin \mathbb{Z}$ and $\frac{2^{m-1} B_{m-1}}{(m-1)!}$ for all $x \in \mathbb{Z}$, where the convergence is pointwise. Moreover,

$$\frac{\tilde{\Omega}_{m+1}}{2} + \sum_{j=1}^{m-2} \frac{2^{j-1}}{j!} \tilde{\Omega}_{m+1-j} \tilde{B}_j(x) + \frac{2^{m-1}}{(m-1)!} \tilde{B}_{m-1}(x)$$

equals $\tilde{\delta}_m(x)$ for all $x \notin \mathbb{Z}$ and

$$\frac{\tilde{\Omega}_{m+1}}{2} + \sum_{j=2}^{m-2} \frac{2^{j-1}}{j!} \tilde{\Omega}_{m+1-j} \tilde{B}_j(x) + \frac{2^{m-1}}{(m-1)!} \tilde{B}_{m-1}(x)$$

equals $\frac{2^{m-1} B_{m-1}}{(m-1)!}$ for all $x \in \mathbb{Z}$.

In [13, 19, 20], it has been shown that

$$\int_0^1 \gamma_m(x) dx = -\frac{2E_{m+1}}{m+1} + \frac{2}{m+1} \sum_{k=1}^{m-1} \sum_{\ell=k+1}^{m} (-1)^{k+\ell} \binom{m+1}{\ell} B_\ell E_{m+1-\ell}.$$

Thus, by Lemma 6, we establish the following identity

$$-\frac{2E_{m+1}}{m+1} + \frac{2}{m+1} \sum_{k=1}^{m-1} \sum_{\ell=k+1}^{m} (-1)^{k+\ell} \binom{m+1}{\ell} B_\ell E_{m+1-\ell} = \frac{\Omega_{m+1}}{2}.$$

Theorems 7, 8, 9 and 10 suggest the following question: For what values of integers $m \geq 1$ does $\Omega_m = 0$ ($\tilde{\Omega}_m = 0$) hold?

4 The functions $\tilde{\eta}_m$ and $\tilde{\theta}_m$

In this section, we consider the functions $\tilde{\eta}_m, (m \geq 2)$ and $\tilde{\theta}_m, (m \geq 2)$ on $\mathbb{R}$, each of which is periodic with period 1. The Fourier series of $\tilde{\eta}_m$ and $\tilde{\theta}_m$ are $\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi inx}$ and $\sum_{n=-\infty}^{\infty} \tilde{C}_n^{(m)} e^{2\pi inx}$, where $C_n^{(m)} = \int_0^1 \tilde{\eta}_m(x) e^{-2\pi inx} dx = \int_0^1 \eta_m(x) e^{-2\pi inx} dx$ and $\tilde{C}_n^{(m)} = \int_0^1 \tilde{\theta}_m(x) e^{-2\pi inx} dx = \int_0^1 \theta_m(x) e^{-2\pi inx} dx$, respectively.

Define $\Delta_m = -2 \sum_{k=1}^{m-1} \frac{B_k E_{m-k}}{k(m-k)} - \frac{E_{m-1}}{m-1}$ and $\tilde{\Delta}_m = -2 \sum_{k=1}^{m-2} \frac{B_k G_{m-k}}{k(m-k)} - \frac{G_{m-1}}{m-1} + \frac{2\delta_{m,2}}{m-1}$, for all $m \geq 2$. To proceed further, we note the following lemma.

Lemma 11. *For $m \geq 2$,*

$$\int_0^1 \eta_m(x) dx = \frac{\Delta_{m+1} + \frac{2}{m(m+1)} E_{m+1}}{m},$$
$$\int_0^1 \theta_m(x) dx = \frac{\tilde{\Delta}_{m+1} + \frac{2}{m(m+1)} G_{m+1}}{m}.$$

Moreover, $\eta_m(1) = \eta_m(0)$ if and only if $\Delta_m = 0$, and $\theta_m(1) = \theta_m(0)$ if and only if $\tilde{\Delta}_m = 0$.

Proof. Recall that $\frac{d}{dx} B_k(x) = k B_{k-1}(x)$ and $\frac{d}{dx} E_k(x) = k E_{k-1}(x)$, for all $k \geq 1$. So, by the definitions, we have

$$\begin{aligned}
\frac{d}{dx} \eta_m(x) &= \sum_{k=1}^{m-1} \frac{k B_{k-1}(x) E_{m-k}(x)}{k(m-k)} + \sum_{k=1}^{m-1} \frac{(m-k) B_k(x) E_{m-1-k}(x)}{k(m-k)} \\
&= (m-1) \sum_{k=1}^{m-2} \frac{B_k(x) E_{m-1-k}(x)}{k(m-1-k)} + \frac{E_{m-1}(x) + B_{m-1}(x)}{m-1} \\
&= (m-1) \eta_{m-1}(x) + \frac{E_{m-1}(x) + B_{m-1}(x)}{m-1}.
\end{aligned}$$

Thus, $\eta_m(x) = \frac{d}{dx}(\eta_{m+1}(x)/m - (E_{m+1}(x) + B_{m+1}(x))/(m^2(m+1)))$, which gives

$$\int_0^1 \eta_m(x) = \frac{\Delta_{m+1} + \frac{2}{m(m+1)} E_{m+1}}{m},$$

as claimed.

Recall that $\frac{d}{dx} G_k(x) = k G_{k-1}(x)$, for all $k \geq 1$. So, by the definitions,

$$\begin{aligned}
\frac{d}{dx} \theta_m(x) &= \sum_{k=1}^{m-1} \frac{k B_{k-1}(x) G_{m-k}(x)}{k(m-k)} + \sum_{k=1}^{m-1} \frac{(m-k) B_k(x) G_{m-1-k}(x)}{k(m-k)} \\
&= (m-1) \sum_{k=1}^{m-2} \frac{B_k(x) G_{m-1-k}(x)}{k(m-1-k)} + \frac{G_{m-1}(x)}{m-1} \\
&= (m-1) \theta_{m-1}(x) + \frac{G_{m-1}(x)}{m-1}.
\end{aligned}$$

Thus, $\theta_m(x) = \frac{d}{dx}(\theta_{m+1}(x)/m - G_{m+1}(x)/(m^2(m+1)))$, which implies

$$\int_0^1 \theta_m(x) = \frac{\tilde{\Delta}_{m+1} + \frac{2}{m(m+1)}G_{m+1}}{m},$$

as claimed. $\square$

Now, we are ready to determine the Fourier coefficients $C_n^{(m)}$ and $\tilde{C}_n^{(m)}$. The case $n = 0$ follows from Lemma 11, that is,

$$C_0^{(m)} = \frac{\Delta_{m+1} + \frac{2}{m(m+1)}E_{m+1}}{m}, \quad \tilde{C}_0^{(m)} = \frac{\tilde{\Delta}_{m+1} + \frac{2}{m(m+1)}G_{m+1}}{m}. \tag{9}$$

Thus, let us assume that $n \neq 0$. By Lemma 11, we have

$$\begin{aligned} C_n^{(m)} &= \int_0^1 \eta_m(x)e^{-2\pi inx}dx = \frac{1}{2\pi in}\int_0^1 \frac{d}{dx}\eta_m(x)e^{-2\pi inx}dx - \frac{1}{2\pi in}\,\eta_m(x)e^{-2\pi inx}\Big|_{x=0}^{x=1} \\ &= \frac{m-1}{2\pi in}\int_0^1 \eta_{m-1}(x)e^{-2\pi inx}dx - \frac{1}{2\pi in}(\eta_m(1)-\eta_m(0)) \\ &+ \frac{1}{2\pi in(m-1)}\int_0^1 (B_{m-1}(x)+E_{m-1}(x))e^{-2\pi inx}dx. \end{aligned}$$

One shows that for $\ell \geq 1$,

$$\int_0^1 B_\ell(x)e^{-2\pi inx}dx = \begin{cases} -\frac{\ell!}{(2\pi in)^\ell}, & n \neq 0, \\ 0, & n = 0, \end{cases}$$

$$\int_0^1 E_\ell(x)e^{-2\pi inx}dx = \begin{cases} 2\sum_{k=1}^{\ell} \frac{(\ell)_{k-1}}{(2\pi in)^k}E_{\ell-k+1}, & n \neq 0, \\ -\frac{2}{\ell+1}E_{\ell+1}, & n = 0, \end{cases}$$

$$\int_0^1 G_\ell(x)e^{-2\pi inx}dx = \begin{cases} 2\sum_{k=1}^{\ell-1} \frac{(\ell)_{k-1}}{(2\pi in)^k}G_{\ell-k+1}, & n \neq 0, \\ -\frac{2}{\ell+1}G_{\ell+1}, & n = 0. \end{cases}$$

Thus,

$$C_n^{(m)} = \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{\Delta_m}{2\pi in} - \frac{(m-2)!}{(2\pi in)^m} + \frac{2}{2\pi in(m-1)}\Phi_m,$$

where $\Phi_m = \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k}E_{m-k}$. Similarly,

$$\tilde{C}_n^{(m)} = \frac{m-1}{2\pi in}\tilde{C}_n^{(m-1)} - \frac{\tilde{\Delta}_m}{2\pi in} + \frac{2}{2\pi in(m-1)}\tilde{\Phi}_m,$$

where $\tilde{\Omega}_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1}}{(2\pi in)^k}G_{m-k}$. Note that $C_n^{(2)} = \int_0^1 (x^2 - x + 1/4)e^{-2\pi inx}dx = -\frac{2}{(2\pi in)^2}$ and $\tilde{C}_n^{(2)} = \int_0^1 (x-1/2)e^{-2\pi inx}dx = -\frac{1}{2\pi in}$. So, by induction on m, we obtain

$$\begin{aligned} C_n^{(m)} &= -\frac{(m-1)!}{(2\pi in)^m} - \sum_{j=1}^{m-2} \frac{(m-1)_{j-1}}{(2\pi in)^j}\Delta_{m+1-j} \\ &- \frac{(m-1)!}{(2\pi in)^m}H_{m-1} + \sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m+1-j}, \end{aligned} \tag{10}$$

where $m \geq 2$. Similarly, one can show that

$$\tilde{C}_n^{(m)} = -\frac{(m-1)!}{(2\pi in)^{m-1}} - \sum_{j=1}^{m-2} \frac{(m-1)_{j-1}}{(2\pi in)^j} \tilde{\Delta}_{m+1-j} + \sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j (m-j)} \tilde{\Phi}_{m+1-j}, \tag{11}$$

for $m \geq 2$.

Lemma 12. *We have*

$$\sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Phi_{m+1-j}$$
$$= \frac{2}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi in)^s} \frac{E_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) - \frac{(m-1)!}{(2\pi in)^m} (H_{m-1} - 1),$$
$$\sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j (m-j)} \tilde{\Phi}_{m+1-j}$$
$$= \frac{2}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi in)^s} \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}).$$

Proof. We present only the proof of the first identity, as that of the second one is analogous. By the definitions, we have

$$\begin{aligned}
\sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Phi_{m+1-j} &= \sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j (m-j)} \sum_{k=1}^{m-j} \frac{(m-1)_{k-1}}{(2\pi in)^k} E_{m-j-k+1} \\
&= \sum_{j=1}^{m-2} \sum_{k=1}^{m-j} \frac{2(m-1)_{j+k-2}}{(2\pi in)^{j+k} (m-j)} E_{m-j-k+1} \\
&= \frac{2}{m} \sum_{j=1}^{m-2} \sum_{k=1}^{m-j} \frac{(m)_{j+k-1}}{(2\pi in)^{j+k} (m-j)} E_{m-j-k+1} \\
&= \frac{2}{m} \sum_{j=1}^{m-2} \sum_{s=j+1}^{m} \frac{(m)_{s-1}}{(2\pi in)^s (m-j)} E_{m-s+1}.
\end{aligned}$$

Thus by interchanging the order of the sums, we obtain

$$\begin{aligned}
\sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Phi_{m+1-j} &= \frac{2}{m} \left\{ \sum_{s=2}^{m-1} \frac{(m)_{s-1}}{(2\pi in)^s} E_{m-s+1} \sum_{j=1}^{s-1} \frac{1}{m-j} + \sum_{j=1}^{m-2} \frac{m!}{(2\pi in)^m (m-j)} E_1 \right\} \\
&= \frac{2}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi in)^s} \frac{E_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) - \frac{(m-1)!}{(2\pi in)^m} (H_{m-1} - 1),
\end{aligned}$$

as required. ∎

Hence, by (10) and (11) with using Lemma 12, we obtain

$$C_n^{(m)} = \frac{-1}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi in)^s}\Delta_{m+1-s} - \frac{2(m-1)!}{(2\pi in)^m}H_{m-1} + \frac{2}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi in)^s}\frac{E_{m-s+1}}{m+1-s}(H_{m-1}-H_{m-s}), \tag{12}$$

$$\tilde{C}_n^{(m)} = \frac{-1}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi in)^s}\tilde{\Delta}_{m+1-s} - \frac{(m-1)!}{(2\pi in)^{m-1}}H_{m-1} + \frac{2}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi in)^s}\frac{G_{m-s+1}}{m+1-s}(H_{m-1}-H_{m-s}), \tag{13}$$

for $m \geq 2$.

Note that the functions $\tilde{\eta}_m$ and $\tilde{\theta}_m(x)$, $m \geq 2$, are piecewise C^∞. Moreover, the functions $\tilde{\eta}_m$ and $\tilde{\theta}_m$ are continuous for those integers $m \geq 2$ with $\Delta_m = 0$ and $\tilde{\Delta}_m = 0$, respectively, and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Delta_m \neq 0$ and $\tilde{\Delta}_m \neq 0$, respectively.

4.1 Case $\Delta_m = 0$ ($\tilde{\Delta}_m = 0$)

Assume first that m is an integer ≥ 2 with $\Delta_m = 0$ ($\tilde{\Delta}_m = 0$). Then $\eta_m(1) = \eta_m(0)$ ($\theta_m(1) = \theta_m(0)$). So, the functions $\tilde{\eta}_m$ and $\tilde{\theta}_m$ are piecewise C^∞ and continuous. Thus, the Fourier series of $\tilde{\eta}_m$ and $\tilde{\theta}_m$ converge uniformly to $\tilde{\eta}_m$ and $\tilde{\theta}_m$, respectively. So, by (9), (12) and (13), we have

$$\begin{aligned}\tilde{\eta}_m(x) &= \frac{\Delta_{m+1}}{m} + \frac{2E_{m+1}}{m^2(m+1)} \\ &+ \sum_{n\in\mathbb{Z}'}\left\{-\frac{2(m-1)!}{(2\pi in)^m}H_{m-1} + \frac{1}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi in)^s}\left(2\frac{E_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \Delta_{m+1-s}\right)\right\}e^{2\pi inx} \\ &= \frac{\Delta_{m+1}}{m} + \frac{2E_{m+1}}{m^2(m+1)} \\ &+ \frac{1}{m}\sum_{s=1}^{m-2}\binom{m}{s}\left(2\frac{E_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \Delta_{m+1-s}\right)s!\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{(2\pi in)^s} \\ &- 2H_{m-1}(m-1)!\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{(2\pi in)^m}\end{aligned}$$

and

$$\begin{aligned}\tilde{\theta}_m(x) &= \frac{\tilde{\Delta}_{m+1}}{m} + \frac{2G_{m+1}}{m^2(m+1)} \\ &+ \sum_{n\in\mathbb{Z}'}\left\{-\frac{(m-1)!}{(2\pi in)^{m-1}}H_{m-1} + \frac{1}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi in)^s}\left(2\frac{G_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \tilde{\Delta}_{m+1-s}\right)\right\}e^{2\pi inx} \\ &= \frac{\tilde{\Delta}_{m+1}}{m} + \frac{2G_{m+1}}{m^2(m+1)} \\ &+ \frac{1}{m}\sum_{s=1}^{m-2}\binom{m}{s}\left(2\frac{G_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \tilde{\Delta}_{m+1-s}\right)s!\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{(2\pi in)^s} \\ &- H_{m-1}(m-1)!\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{(2\pi in)^{m-1}}.\end{aligned}$$

Thus, by (1) and (2), we obtain

$$
\begin{aligned}
\tilde{\eta}_m(x) &= \frac{\Delta_{m+1}}{m} + \frac{2E_{m+1}}{m^2(m+1)} \\
&\quad - \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{2E_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \Delta_{m+1-s}\right)\tilde{B}_s(x) + \frac{2}{m}H_{m-1}\tilde{B}_m(x) \\
&\quad + \Delta_m \begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}, \end{cases} \\
\tilde{\theta}_m(x) &= \frac{\tilde{\Delta}_{m+1}}{m} + \frac{2G_{m+1}}{m^2(m+1)} \\
&\quad - \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{2G_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \tilde{\Delta}_{m+1-s}\right)\tilde{B}_s(x) + H_{m-1}\tilde{B}_{m-1}(x) \\
&\quad + \tilde{\Delta}_m \begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}, \end{cases}
\end{aligned}
$$

for all $x \in \mathbb{R}$. Thus, we can state the following results.

Theorem 13. *Let $m \geq 2$ be an integer with $\Delta_m = 0$. Then the function $\tilde{\eta}_m(x)$ has the Fourier series expansion*

$$
\begin{aligned}
\tilde{\eta}_m(x) &= \frac{\Delta_{m+1}}{m} + \frac{2E_{m+1}}{m^2(m+1)} \\
&\quad + \sum_{n\in\mathbb{Z}'}\left\{-\frac{2(m-1)!}{(2\pi in)^m}H_{m-1} + \frac{1}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi in)^s}\left(2\frac{E_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \Delta_{m+1-s}\right)\right\}e^{2\pi inx},
\end{aligned}
$$

for all $x \in \mathbb{R}$, where the convergence is uniform. Moreover,

$$
\begin{aligned}
\tilde{\eta}_m(x) &= \frac{\Delta_{m+1}}{m} + \frac{2E_{m+1}}{m^2(m+1)} \\
&\quad - \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{2E_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \Delta_{m+1-s}\right)\tilde{B}_s(x) + \frac{2}{m}H_{m-1}\tilde{B}_m(x),
\end{aligned}
$$

for all $x \in \mathbb{R}$.

Theorem 14. *Let $m \geq 2$ be an integer with $\tilde{\Delta}_m = 0$. Then the function $\tilde{\theta}_m(x)$ has the Fourier series expansion*

$$
\begin{aligned}
\tilde{\theta}_m(x) &= \frac{\tilde{\Delta}_{m+1}}{m} + \frac{2G_{m+1}}{m^2(m+1)} \\
&\quad + \sum_{n\in\mathbb{Z}'}\left\{-\frac{(m-1)!}{(2\pi in)^m}H_{m-1} + \frac{1}{m}\sum_{s=1}^{m-2}\frac{(m)_s}{(2\pi in)^s}\left(2\frac{G_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \tilde{\Delta}_{m+1-s}\right)\right\}e^{2\pi inx},
\end{aligned}
$$

for all $x \in \mathbb{R}$, where the convergence is uniform. Moreover,

$$
\begin{aligned}
\tilde{\theta}_m(x) &= \frac{\tilde{\Delta}_{m+1}}{m} + \frac{2G_{m+1}}{m^2(m+1)} \\
&\quad - \frac{1}{m}\sum_{s=2}^{m-2}\binom{m}{s}\left(\frac{2G_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \tilde{\Delta}_{m+1-s}\right)\tilde{B}_s(x) + H_{m-1}\tilde{B}_{m-1}(x),
\end{aligned}
$$

for all $x \in \mathbb{R}$.

4.2 Case $\Delta_m \neq 0$ ($\tilde{\Delta}_m \neq 0$)

Assume next that m is an integer ≥ 2 with $\Delta_m \neq 0$ ($\tilde{\Delta}_m \neq 0$). Then $\eta_m(1) \neq \eta_m(0)$ ($\theta(1) \neq \theta(0)$). So, the functions $\tilde{\eta}_m$ and $\tilde{\theta}_m$ are piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus, the Fourier series of $\tilde{\eta}_m(x)$ and $\tilde{\theta}_m$ converge pointwise to $\tilde{\eta}_m(x)$ and $\tilde{\theta}_m$ for all $x \notin \mathbb{Z}$, and converge to

$$\frac{\eta_m(1) + \eta_m(0)}{2} = -\frac{E_{m-1}}{2(m-1)},$$
$$\frac{\theta_m(1) + \theta_m(0)}{2} = \frac{2^{m-1}}{m-1} B_{m-1},$$

for all $x \in \mathbb{Z}$. Then, by Theorems 13 and 14, we obtain the following results.

Theorem 15. *Let $m \geq 2$ be an integer with $\Lambda_m \neq 0$. Then*

$$\frac{\Delta_{m+1}}{m} + \frac{2E_{m+1}}{m^2(m+1)}$$
$$+ \sum_{n \in \mathbb{Z}'} \left\{ -\frac{2(m-1)!}{(2\pi in)^m} H_{m-1} + \frac{1}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi in)^s} \left(2\frac{E_{m+1-s}}{m+1-s}(H_{m-1} - H_{m-s}) - \Delta_{m+1-s} \right) \right\} e^{2\pi inx},$$

equals $\tilde{\eta}_m(x)$ for all $x \notin \mathbb{Z}$ and $-\frac{E_{m-1}}{2(m-1)}$ for all $x \in \mathbb{Z}$, where the convergence is pointwise. Moreover,

$$\frac{\Delta_{m+1}}{m} + \frac{2E_{m+1}}{m^2(m+1)}$$
$$- \frac{1}{m} \sum_{s=1}^{m-1} \binom{m}{s} \left(\frac{2E_{m+1-s}}{m+1-s}(H_{m-1} - H_{m-s}) - \Delta_{m+1-s} \right) \tilde{B}_s(x) + \frac{2}{m} H_{m-1} \tilde{B}_m(x),$$

equals $\tilde{\eta}_m(x)$ for all $x \notin \mathbb{Z}$ and

$$\frac{\Delta_{m+1}}{m} + \frac{2E_{m+1}}{m^2(m+1)}$$
$$- \frac{1}{m} \sum_{s=2}^{m-1} \binom{m}{s} \left(\frac{2E_{m+1-s}}{m+1-s}(H_{m-1} - H_{m-s}) - \Delta_{m+1-s} \right) \tilde{B}_s(x) + \frac{2}{m} H_{m-1} \tilde{B}_m(x),$$

equals $-\frac{E_{m-1}}{2(m-1)}$ for all $x \in \mathbb{Z}$.

Theorem 16. *Let $m \geq 2$ be an integer with $\tilde{\Delta}_m \neq 0$. Then*

$$\frac{\tilde{\Delta}_{m+1}}{m} + \frac{2G_{m+1}}{m^2(m+1)}$$
$$+ \sum_{n \in \mathbb{Z}'} \left\{ -\frac{(m-1)!}{(2\pi in)^m} H_{m-1} + \frac{1}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi in)^s} \left(2\frac{G_{m+1-s}}{m+1-s}(H_{m-1} - H_{m-s}) - \tilde{\Delta}_{m+1-s} \right) \right\} e^{2\pi inx},$$

equals $\tilde{\theta}_m(x)$ for all $x \notin \mathbb{Z}$ and $\frac{2^{m-1}B_{m-1}}{m-1}$ for all $x \in \mathbb{Z}$, where the convergence is piecewise. Moreover,

$$\frac{\tilde{\Delta}_{m+1}}{m} + \frac{2G_{m+1}}{m^2(m+1)}$$
$$- \frac{1}{m} \sum_{s=1}^{m-2} \binom{m}{s} \left(\frac{2G_{m+1-s}}{m+1-s}(H_{m-1} - H_{m-s}) - \tilde{\Delta}_{m+1-s} \right) \tilde{B}_s(x) + H_{m-1} \tilde{B}_{m-1}(x)$$

equals $\tilde{\theta}_m(x)$ *for all* $x \notin \mathbb{Z}$ *and*

$$\frac{\tilde{\Delta}_{m+1}}{m} + \frac{2G_{m+1}}{m^2(m+1)} - \frac{1}{m}\sum_{s=1}^{m-2}\binom{m}{s}\left(\frac{2G_{m+1-s}}{m+1-s}(H_{m-1}-H_{m-s}) - \tilde{\Delta}_{m+1-s}\right)\tilde{B}_s(x) + H_{m-1}\tilde{B}_{m-1}(x)$$

equals $\frac{2^{m-1}B_{m-1}}{m-1}$ *for all* $x \in \mathbb{Z}$.

In [13, 19, 20] has been shown that

$$\int_0^1 \eta_m(x)dx = \frac{2}{m(m^2-1)}\sum_{k=0}^{m-2}\sum_{\ell=k+2}^{m}(-1)^{k+\ell+1}\frac{\binom{m+1}{\ell}}{\binom{m-2}{k}}B_\ell E_{m+1-\ell}.$$

Thus, by Lemma 11, we establish the following identity

$$\frac{2}{m(m^2-1)}\sum_{k=0}^{m-2}\sum_{\ell=k+2}^{m}(-1)^{k+\ell+1}\frac{\binom{m+1}{\ell}}{\binom{m-2}{k}}B_\ell E_{m+1-\ell} = \frac{\Delta_{m+1}}{m} + \frac{2E_{m+1}}{m^2(m+1)}.$$

Theorems 13, 14, 15 and 16 suggest the following question: For what values of integers $m \geq 2$ does $\Delta_m = 0$ ($\tilde{\Delta}_m = 0$) hold?

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