A new proof of Euler’s pentagonal number theorem

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Abstract: A new proof of Euler’s pentagonal number theorem is obtained.
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1 History and motivation

The classical statement of Euler’s pentagonal number theorem is

\[
\prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} q^{\frac{n(3n-1)}{2}}, \text{ where } |q| < 1. \tag{1.1}
\]

By expanding the left side of the equation (1.1), one can see that

\[
\prod_{n=1}^{\infty} (1 - q^n) = 1 + \sum_{n=1}^{\infty} (r_e(n) - r_o(n)) q^n, \tag{1.2}
\]

where \(r_e(n)\) denotes the number of distinct partitions (partitions with distinct parts) of \(n\) with even number of parts, and \(r_o(n)\) denotes the number of distinct partitions of \(n\) with odd number of parts.

Equations (1.1) and (1.2) together give the following expression:

\[
r_e(n) - r_o(n) = \begin{cases} (-1)^k, & \text{if } n = \frac{3k^2 + k}{2}; \\ 0, & \text{otherwise}. \end{cases} \tag{1.3}
\]
This expression is known as the partition-theoretic interpretation of Euler’s pentagonal number theorem. Euler’s pentagonal number theorem follows directly from the Jacobi’s triple product identity
\[
\prod_{m=1}^{\infty} (1 - q^{2m}) (1 + q^{2m-1} z^2) (1 + q^{2m-1} z^{-2}) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n}
\]
for \( q = x^{\frac{3}{2}} \) and \( z^2 = -x^{\frac{1}{2}} \).


Euler [5] proved the classical version of his theorem using induction. Many mathematicians obtained proofs for Jacobi’s triple product identity (for proof see [1, 2, 3, 9, 11]). Addition to these proofs, Franklin [7] gave a bijective proof for Euler’s pentagonal number theorem using Ferrer’s diagram of the partition, and F. J. Dyson [4] gave a combinatorial proof involving the idea of the rank of a partition.

In this article, we give a new proof for the partition-theoretic version of Euler’s pentagonal number theorem.

**Definition 1.1.** Let \( n \) be a positive integer. A partition \((a_1, a_2, \ldots, a_k)\) of \( n \) is said to be a distinct partition of \( n \) if \( a_i > a_{i+1} \) for every \( i \in \{1, 2, \ldots, k-1\} \).

## 2 Proof

Let \( n \) be a positive integer. Let \( Q_n \) be the set of all distinct partitions of \( n \). Define an operator \( \phi : Q_n \to Q_n \) by
\[
\phi((a_1, a_2, \ldots, a_k)) = (a_1 + 1, a_2 + 1, \ldots, a_{a_k} + 1, a_{a_k+1}, \ldots, a_{k-1})
\]
when \( a_k < k \).

Let \( Q_{n,s} \) be the set of all distinct partitions of \( n \) with its least part \( s \) such that \( s < \) number of parts.

Put \( A_1 = Q_{n,1} \). Define \( \phi : A_1 \to Q_n \). Since every partition in \( \phi(A_1) \) has least part greater than 1, we have \( \phi(A_1) \cap A_1 = \emptyset \). Since each partition in \( A_1 \) has identical least part, \( \phi \) cannot be a many-to-one mapping. Thus, \( \phi \) is an one-to-one mapping. Moreover, we see that image of every partition with even (resp. odd) number of parts in \( A_1 \) under \( \phi \) has odd (resp. even) number of parts. Consequently, the number of even partitions (partitions with even number of parts) and odd partitions (partitions with odd number of parts) in \( \phi(A_1) \cup A_1 \) are same.

Define \( A_2 = (Q_n \setminus (A_1 \cup \phi(A_1))) \cap Q_{n,2} \). Consider the mapping \( \phi : A_2 \to Q_n \). Following the line of argument in the last paragraph, we again get that \( \phi(A_2) \cap A_2 = \emptyset \) and the number of even partitions and odd partitions in \( \phi(A_2) \cup A_2 \) are same.

For \( k \geq 3 \), define \( A_k = (Q_n \setminus \bigcup_{i=1}^{k-1} (A_i \cup \phi(A_i))) \cap Q_{n,k} \). We see that there is no possibility for the existence of a distinct partition say \( \pi_2 \) such that \( \pi_2 \in A_r \) and \( \phi(\pi_2) \in \phi(A_l) \) for some
\( l < r \). For otherwise, there will be a distinct partition say \( \pi_1 \) such that \( \phi(\pi_1) = \phi(\pi_2) \) with \( \pi_1 \neq \pi_2 \). This gives \((a_1 + 1, a_2 + 1, \ldots, a_l + 1, a_{l+1}, \ldots, a_{k-1}) = (b_1 + 1, b_2 + 1, \ldots, b_l + 1, b_{l+1} + 1, \ldots, b_r + 1, b_{r+1}, \ldots, b_{k-1})\), where \( \pi_1 = (a_1, \ldots, a_l) \) and \( \pi_2 = (b_1, \ldots, b_k) \) with \( a_k = l \) and \( b_k = r \). Consider the partition \( \pi^* = (b_1, b_2, \ldots, b_l + 1, \ldots, b_r + 1, b_{r+1}, \ldots, b_{k-1}, b_k, l) \).

From the above equality we have \( b_{l+1} + 1 < b_l \) and since \( l < b_k < k \), one can see that \( \pi^* \) is a distinct partition of \( n \) with least part \( l \) such that \( l \) is less than \( k \). Furthermore, \( \phi(\pi^*) = \pi_2 \). Thus, \( \pi_2 \in \phi(A_l) \) which implies \( \pi_2 \notin A_r \), which is a contradiction.

Accordingly, we have the following conclusions:

1. \( \phi(A_k) \cap A_k = \emptyset \) for every \( k \in \{1, 2, \ldots\} \).

2. The number of even and odd partitions in \( \cup_{i \geq 1} (A_i \cup \phi(A_i)) \) are same.

Let \( Q_n^* = \cup_{i \geq 1} (A_i \cup \phi(A_i)) \). A closer examination of the set \( Q_n \setminus Q_n^* \) completes the proof. Let \( \pi = (a_1, a_2, \ldots, a_l) \in Q_n \setminus Q_n^* \). Define \( c(\pi) \) to be the largest integer \( l \geq 2 \) for which \( a_1, a_2, \ldots, a_l \) satisfies \( a_2 - a_1 = a_3 - a_2 = \cdots = a_l - a_{l-1} = 1 \). We claim that \( c(\pi) = k \). If \( c(\pi) = s \) for some \( s < k \), then it is plain that we can write \( \pi = (b, b - 1, \ldots, b - (s - 1), a_{s+1}, \ldots, a_k) \) with \((b - (s - 1)) - a_{s+1} = 1\). Now consider the partition \( \pi_1 = (b - 1, b - 2, \ldots, b - s, a_{s+1}, \ldots, a_k) \) from the membership of \( \pi \), we have \( a_k \geq k \). Since \( s < k \), we have \( a_k - s > 0 \). Thus \( \pi_1 \) is a distinct partition of \( n \). Also, we have \( \phi(\pi_1) = \pi \). If \( \pi_1 \in A_i \) for some \( i \), then we have \( \pi \in \phi(A_i) \), which leads to the conclusion that \( \pi \in Q_n^* \) which is a contradiction. On the other hand, if \( \pi_1 \in \phi(A_j) \) for some \( j \), then there exist a distinct partition say \( \pi_2 = (b_1, b_2, \ldots, b_{k+2}) \) such that \( \phi(\pi_2) = \pi_1 \). Note that \( 1 \leq b_{k+2} < s \). From this it follows that \( 1 \leq b_{k+2} < k \) and \( b_{k+2} < s \). Since \( \phi(\pi_2) = \pi_1 \), we have the following equalities: \( b_1 + 1 = b - 1, b_2 + 1 = b - 2, \ldots, b_{k+2} + 1 = b - b_{k+2}, b_{k+2} + 1 = b - b_{k+2} - 1, \ldots; \) which leads to the equality \( b_{k+2} - b_{k+2} + 1 = 0 \) which is a contradiction. Thus \( c(\pi) = k \). Accordingly, \( \pi \) is of the form \( \pi = (a_k + k - 1, a_k + k - 2 + 1, \ldots, a_k + 1, a_k) \).

We claim that \( a_k \) can assume only two values namely \( k \) or \( k + 1 \). From the membership of \( \pi \) it follows that \( a_k \geq k \). Suppose that \( a_k > k + 1 \). Then consider the partition \( \pi_1 = (a_k + (k - 2), \ldots, a_k, a_k - 1, k) \). Clearly, \( \pi_1 \) is a distinct partition of \( n \). We see that \( \phi(\pi_1) = \pi \), which implies that \( \pi_1 \notin Q_n \setminus Q_n^* \). This in turn implies that \( \pi_1 \notin Q_n^* \). If \( \pi_1 \in A_i \) for some \( i \), then we would have \( \phi(\pi_1) \in \phi(A_i) \), that is, \( \pi \in Q_n^* \) which is a contradiction. If \( \pi_1 \in \phi(A_j) \) for some \( j \), then there will be a distinct partition of \( n \) say \( \pi_2 = (b_1, b_2, \ldots, b_{k+2}) \) such that \( \phi(\pi_2) = \pi_1 \). Now we make it a point that \( b_{k+2} < k \). Since \( \phi(\pi_2) = \pi_1 \), we have the equalities \( b_1 + 1 = a_k + (k - 2), b_2 + 1 = a_k + (k - 3), \ldots, b_{k+2} + 1 = a_k + (k - 1) - b_{k+2}, b_{k+2} + 1 = a_k + (k - 1) - (b_{k+2} + 1), \ldots; \) this gives \( b_{k+2} = b_{k+2} + 1 \), which is absurd. Thus the claim follows.

From these observations, we get that \( r_e(n) - r_o(n) = 0 \) if \( n \) is not of the forms: \( k + (k + 1) + \cdots + (k + (k - 1)) \) and \( (k + 1) + (k + 2) + \cdots + (k + k) \), that is, when \( n \neq \frac{3k^2 + k}{2} \). On the other hand, if \( n = \frac{3k^2 + k}{2} \) then we have \( r_e(n) - r_o(n) = 1 \) when \( k \) is even, and \( r_e(n) - r_o(n) = -1 \) when \( k \) is odd.

This completes the proof. \( \square \)
References


