On Dris conjecture about odd perfect numbers

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Abstract: The Euler’s form of odd perfect numbers, if any, is \( n = \pi^\alpha N^2 \), where \( \pi \) is prime, \((\pi, N) = 1\) and \( \pi \equiv \alpha \equiv 1 \pmod{4} \). Dris conjecture states that \( N > \pi^\alpha \). We find that \( N^2 > \frac{1}{2}\pi\gamma \), with \( \gamma = \max\{\omega(n) - 1, \alpha\} \); \( \omega(n) \geq 10 \) is the number of distinct prime factors of \( n \).

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1 Introduction

Without explicit definitions all the numbers considered in what follows must be taken as strictly positive integers. Let \( \sigma(n) \) be the sum of the divisors of a number \( n \); \( n \) is said to be perfect if and only if \( \sigma(n) = 2n \). The multiplicative structure of odd perfect numbers, if any, is

\[
n = \pi^\alpha N^2, \tag{1}
\]

where \( \pi \) is prime, \( \pi \equiv \alpha \equiv 1 \pmod{4} \) and \((\pi, N) = 1\) (Euler, cited in [3, p. 19]); \( \pi^\alpha \) is called the Euler’s factor. From equation (1) and from the fact that the \( \sigma \) is multiplicative, it results also

\[
n = \frac{\sigma(\pi^\alpha)}{2} \sigma(N^2), \tag{2}
\]

where \( \sigma(N^2) \) is odd and \( 2\|\sigma(\pi^\alpha) \). Many details concerning the Euler’s factor and \( N^2 \) are given, for example, in [2, 5, 8, 9, 10].

Regarding the relation between the magnitude of \( N^2 \) and \( \pi^\alpha \) it has been conjectured by Dris that \( N > \pi^\alpha \) [4]. The result obtained in this paper is a necessary condition for odd perfection (Theorem 2.1) which provides an indication about Dris conjecture.
Indicating with $\omega(n)$ the number of distinct prime factors of $n$, we prove that (Corollary 2.3):

\[(i) \quad N^2 > \frac{1}{2} \pi \gamma, \text{ where } \gamma = \max \{\omega(n) - 1, \alpha\}.\]

Since $\omega(n) \geq 10$ (Nielsen, [6]), it follows:

\[(i)_1 \quad N^2 > \frac{1}{2} \pi^9; \text{ this improves the result } N > \pi \text{ claimed in [1] by Brown in his approach to Dris conjecture.}\]

Besides

\[(i)_2 \quad \text{If } \omega(n) - 1 > 2\alpha, \text{ then } N > \pi^\alpha,\]

so that

\[(i)_3 \quad \text{If } \omega(n) - 1 > 2\alpha \text{ for each odd perfect number } n, \text{ then Dris conjecture is true.}\]

Now, some questions arise: $\omega(n)$ depends on $\alpha$? Is there a maximum value of $\alpha$? The minimum value of $\alpha$ is 1? The only possible value of $\alpha$ is 1 (Sorli, [7, conjecture 2]) so that Dris conjecture is true? Without ever forgetting the main question: do odd perfect numbers exist?

## 2 The proof

Referring to an odd perfect number $n$ with the symbols used in equation (1), we obtain:

**Lemma 2.1.** If $n$ is an odd perfect number, then

\[N^2 = A \frac{\sigma(\pi^{\alpha})}{2} \quad \text{and} \quad \sigma(N^2) = A \pi^{\alpha}.\]

**Proof.** From equation (2) and from the fact that $(\sigma(\pi^{\alpha}), \pi^{\alpha}) = 1$, it follows

\[N^2 = A \frac{\sigma(\pi^{\alpha})}{2}, \tag{3}\]

where $A$ is an odd positive integer given by

\[A = \frac{\sigma(N^2)}{\pi^{\alpha}}. \tag{4}\]

In relation to the odd parameter $A$ in Lemma 2.1, we give two further lemmas:

**Lemma 2.2.** If $A = 1$, then $\alpha \geq \omega(n) - 1$ and $N^2 > \frac{1}{2} \pi^{\alpha}$.

**Proof.** Let $q_k, k = 1, 2, ..., \omega(N) = \omega(N^2)$, are the prime factors of $N^2$; from hypothesis and from (4) we have...
\[ \pi^\alpha = \sigma(N^2) = \sigma\left( \prod_{k=1}^{\omega(N)} \frac{q_k^{2^3k}}{2^3} \right) = \prod_{k=1}^{\omega(N)} \frac{\sigma(q_k^{2^3k})}{2^3} = \prod_{k=1}^{\omega(N)} \delta_k \]

in which \( \alpha = \sum_{k=1}^{\omega(N)} \delta_k \geq \sum_{k=1}^{\omega(N)} 1 = \omega(N) \).

Since \( \omega(n) = \omega(N) + 1 \), it results in
\[ \alpha \geq \omega(n) - 1. \]

Besides, from Equation (3) it follows
\[ N^2 = \frac{1}{2} \sigma(\pi^\alpha) > \frac{1}{2} \pi^\alpha. \]

**Lemma 2.3.** If \( A > 1 \), then \( N^2 > \frac{3}{2} \pi^\alpha \).

**Proof.** From Equation (3) it results \( A \geq 3 \). Thus
\[ N^2 \geq \frac{3}{2} \sigma(\pi^\alpha) > \frac{3}{2} \pi^\alpha. \]

The following theorem summarizes a necessary condition for odd perfection.

**Theorem 2.1.** If \( n \) is an odd perfect number, then
\[ (\neg a \land d) \lor (a \land b \land c) \lor (b \land c \land d), \]
where: \( a \equiv (A = 1), \neg a \equiv (A > 1) \), \( b \equiv (\alpha \geq \omega(n) - 1) \), \( c \equiv (N^2 > \frac{1}{2} \pi^\alpha) \), \( d \equiv (N^2 > \frac{3}{2} \pi^\alpha) \).

**Proof.** We combine Lemmas 2.2 and 2.3, setting
\[
\begin{align*}
\text{lemma 2.2 : } & (a \implies b \land c) \text{,} \\
\text{lemma 2.3 : } & (\neg a \implies d) \text{.}
\end{align*}
\]
where, since it cannot be \( A < 1 \), it is \( (a) \equiv (A = 1) \) and \( (\neg a) \equiv (A > 1) \). One obtains from (5)
\[ [\neg a \lor (b \land c)] \land (a \lor d), \]
which is equivalent to
\[ (\neg a \land d) \lor (a \land b \land c) \lor (b \land c \land d). \]

Considering cases in which the necessary condition for odd perfection (6) is false, we obtain the following corollaries:

**Corollary 2.1.** If \( n \) is an odd perfect number, then \( N^2 > \frac{1}{2} \pi^\alpha \).

**Proof.** We have
\[ (\neg c \land \neg d)(\equiv N^2 < \frac{1}{2} \pi^\alpha) \implies n \text{ is not an odd perfect number.} \]

From the contrapositive formulation of (7) it follows the proof.
Corollary 2.2. If $n$ is an odd perfect number, then
\[ N^2 > \frac{3}{2} \pi^{\omega(n)-1} > \frac{1}{2} \pi^{\omega(n)-1}. \]

Proof. We have
\[ (\neg b \land \neg d)(\equiv N^2 < \frac{3}{2} \pi^{\omega(n)-1}) \implies n \text{ is not an odd perfect number.} \]

From the contrapositive formulation of (8) it follows the proof.

Combining these two corollaries, we have

Corollary 2.3. If $n$ is an odd perfect number, then
\[ N^2 > \frac{1}{2} \pi^\gamma, \text{ where } \gamma = \max\{\omega(n) - 1, \alpha\}. \]

Proof. Immediate.

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References


