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On Dris conjecture about odd perfect numbers

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Abstract: The Euler's form of odd perfect numbers, if any, is $n = \pi^\alpha N^2$, where π is prime, $(\pi, N) = 1$ and $\pi \equiv \alpha \equiv 1 \pmod 4$. Dris conjecture states that $N > \pi^\alpha$. We find that $N^2 > \frac{1}{2}\pi^\gamma$, with $\gamma = max\{\omega(n) - 1, \alpha\}$; $\omega(n) \geq 10$ is the number of distinct prime factors of n.
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1 Introduction

Without explicit definitions all the numbers considered in what follows must be taken as strictly positive integers. Let $\sigma(n)$ be the sum of the divisors of a number n; n is said to be perfect if and only if $\sigma(n) = 2n$. The multiplicative structure of odd perfect numbers, if any, is

$$n = \pi^\alpha N^2, \tag{1}$$

where π is prime, $\pi \equiv \alpha \equiv 1 \pmod 4$ and $(\pi, N) = 1$ (Euler, cited in [3, p. 19]); π^α is called the Euler's factor. From equation (1) and from the fact that the σ is multiplicative, it results also

$$n = \frac{\sigma(\pi^\alpha)}{2}\sigma(N^2), \tag{2}$$

where $\sigma(N^2)$ is odd and $2\|\sigma(\pi^\alpha)$. Many details concerning the Euler's factor and N^2 are given, for example, in [2, 5, 8, 9, 10].

Regarding the relation between the magnitudo of N^2 and π^α it has been conjectured by Dris that $N > \pi^\alpha$ [4]. The result obtained in this paper is *a necessary condition for odd perfection* (Theorem 2.1) which provides an indication about Dris conjecture.

Indicating with $\omega(n)$ the number of distinct prime factors of n, we prove that (Corollary 2.3):

(i) $N^2 > \frac{1}{2}\pi^{\gamma}$, where $\gamma = max\{\omega(n)-1, \alpha\}$.

Since $\omega(n) \geq 10$ (Nielsen, [6]), it follows:

$(i)_1$ $N^2 > \frac{1}{2}\pi^9$; this improves the result $N > \pi$ claimed in [1] by Brown in his approach to Dris conjecture.

Besides

$(i)_2$ If $\omega(n) - 1 > 2\alpha$, then $N > \pi^{\alpha}$,

so that

$(i)_3$ If $\omega(n) - 1 > 2\alpha$ for each odd perfect number n, then Dris conjecture is true.

Now, some questions arise: $\omega(n)$ depends on α? Is there a maximum value of α? The minimum value of α is 1? The only possible value of α is 1 (Sorli, [7, conjecture 2]) so that Dris conjecture is true? Without ever forgetting the main question: do odd perfect numbers exist?

2 The proof

Referring to an odd perfect number n with the symbols used in equation (1), we obtain:

Lemma 2.1. *If n is an odd perfect number, then*

$$N^2 = A\frac{\sigma(\pi^{\alpha})}{2} \text{ and } \sigma(N^2) = A\pi^{\alpha}.$$

Proof. From equation (2) and from the fact that $(\sigma(\pi^{\alpha}), \pi^{\alpha}) = 1$, it follows

$$N^2 = A\frac{\sigma(\pi^{\alpha})}{2}, \tag{3}$$

where A is an odd positive integer given by

$$A = \frac{\sigma(N^2)}{\pi^{\alpha}}. \tag{4}$$

□

In relation to the odd parameter A in Lemma 2.1, we give two further lemmas:

Lemma 2.2. *If $A = 1$, then $\alpha \geq \omega(n) - 1$ and $N^2 > \frac{1}{2}\pi^{\alpha}$.*

Proof. Let $q_k, k = 1, 2, ..., \omega(N) = \omega(N^2)$, are the prime factors of N^2; from hypothesis and from (4) we have

$$\pi^\alpha = \sigma(N^2) = \sigma(\prod_{k=1}^{\omega(N)} q_k^{2\beta_k}) = \prod_{k=1}^{\omega(N)} \sigma(q_k^{2\beta_k}) = \prod_{k=1}^{\omega(N)} \pi^{\delta_k}$$

in which $\alpha = \sum_{k=1}^{\omega(N)} \delta_k \geq \sum_{k=1}^{\omega(N)} 1_k = \omega(N)$.

Since $\omega(n) = \omega(N) + 1$, it results in

$$\alpha \geq \omega(n) - 1.$$

Besides, from Equation (3) it follows

$$N^2 = \frac{1}{2}\sigma(\pi^\alpha) > \frac{1}{2}\pi^\alpha. \qquad \square$$

Lemma 2.3. *If $A > 1$, then $N^2 > \frac{3}{2}\pi^\alpha$.*

Proof. From Equation (3) it results $A \geq 3$. Thus

$$N^2 \geq \frac{3}{2}\sigma(\pi^\alpha) > \frac{3}{2}\pi^\alpha. \qquad \square$$

The following theorem summarizes a necessary condition for odd perfection.

Theorem 2.1. *If n is an odd perfect number, then*

$$(\neg a \wedge d) \vee (a \wedge b \wedge c) \vee (b \wedge c \wedge d),$$

where: $a \cong (A = 1)$,$\neg a \cong (A > 1)$, $b \cong (\alpha \geq \omega(n) - 1)$, $c \cong (N^2 > \frac{1}{2}\pi^\alpha)$, $d \cong (N^2 > \frac{3}{2}\pi^\alpha)$.

Proof. We combine Lemmas 2.2 and 2.3, setting

$$\begin{cases} lemma\ 2.2 : (a \implies b \wedge c) \\ lemma\ 2.3 : (\neg a \implies d) \end{cases}, \qquad (5)$$

where, since it cannot be $A < 1$, it is $(a) \cong (A = 1)$ and $(\neg a) \cong (A > 1)$. One obtains from (5)

$$[\neg a \vee (b \wedge c)] \wedge (a \vee d),$$

which is equivalent to

$$(\neg a \wedge d) \vee (a \wedge b \wedge c) \vee (b \wedge c \wedge d). \qquad (6)$$

$\square$

Considering cases in which the necessary condition for odd perfection (6) is false, we obtain the following corollaries:

Corollary 2.1. *If n is an odd perfect number, then $N^2 > \frac{1}{2}\pi^\alpha$.*

Proof. We have

(7) $(\neg c \wedge \neg d)(\cong\ N^2 < \frac{1}{2}\pi^\alpha) \implies n$ is not an odd perfect number.

From the contrapositive formulation of (7) it follows the proof. $\square$

Corollary 2.2. *If n is an odd perfect number, then*

$$N^2 > \frac{3}{2}\pi^{\omega(n)-1} > \frac{1}{2}\pi^{\omega(n)-1}.$$

Proof. We have

(8) $(\neg b \wedge \neg d)(\cong\ N^2 < \frac{3}{2}\pi^{\omega(n)-1}) \implies n$ is not an odd perfect number.

From the contrapositive formulation of (8) it follows the proof. □

Combining these two corollaries, we have

Corollary 2.3. *If n is an odd perfect number, then*

$$N^2 > \frac{1}{2}\pi^{\gamma}, where\ \gamma = max\{\omega(n) - 1, \alpha\}.$$

Proof. Immediate. □

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