Generalized dual Pell quaternions

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Abstract: In this paper, we defined the generalized dual Pell quaternions. Also, we investigated the relations between the generalized dual Pell quaternions. Furthermore, we gave the Binet’s formulas and Cassini-like identities for these quaternions.

Keywords: Pell number, Pell quaternion, Lucas quaternion, Dual quaternion.

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1 Introduction

The real quaternions are a number system which extends to the complex numbers. They are first described by Irish mathematician William Rowan Hamilton in 1843.

Hamilton [1] introduced the set of real quaternions which can be represented as

\[ H = \{ q = q_0 + i q_1 + j q_2 + k q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \} \]  

(1.1)

where

\[ i^2 = j^2 = k^2 = -1, \quad j i = -j i = k, \quad j k = -k j = i, \quad k i = -i k = j. \]
Several authors worked on different quaternions and their generalizations. ([2–22, 24–26]). In 2013, Akıyiğit et al. [17] defined split Fibonacci and split Lucas quaternions and obtained some identities for them. Complex split quaternions were defined by Kula and Yayli [13] in 2007.

In 1961, Horadam [3] firstly introduced the generalized Fibonacci sequence \( (H_n) \) and used this sequence in 1963, Horadam [4] defined the \( n \)-th Fibonacci quaternion which can be represented as

\[
Q_F = \{ Q_n = F_n + i F_{n+1} + j F_{n+2} + k F_{n+3} \mid F_n, \ n-th \ Fibonacci \ number \} \quad (1.2)
\]

where

\[
i^2 = j^2 = k^2 = i j k = -1, \quad i j = -j i = k, \quad j k = -k j = i,
\]

and \( n \geq 1 \).


In 1973, Swamy [8] considered generalized Fibonacci quaternions as a new quaternion as follows:

\[
P_n = H_n + i H_{n+1} + j H_{n+2} + k H_{n+3} \quad (1.3)
\]

where

\[
\begin{aligned}
H_n & = H_{n-1} + H_{n-2}, \\
H_1 & = p, \\
H_2 & = p + q,
\end{aligned}
\]

\[
H_n = (p - q)F_n + qF_{n+1}, \ n \geq 1
\]

where \( H_n \) is the \( n-th \) generalized Fibonacci number that is defined in [4].

(See [8] for generalized Fibonacci quaternions).

In 1977, Iakin [9, 10] introduced higher order quaternions and gave some identities for these quaternions.


In 2006, Majernik [18] defined dual quaternions as follows:

\[
H_D = \left\{ Q = a + b i + c j + d k \mid a, b, c, d \in \mathbb{R}, \ i^2 = j^2 = k^2 = i j k = 0, \ i j = -j i = k, \ j k = -k j = i, \ k i = -i k = 0 \right\}. \quad (1.4)
\]

In 2009, Ata and Yaylı [14] defined dual quaternions with dual numbers coefficient \((a + \varepsilon b, \ a, b \in \mathbb{R}, \ \varepsilon^2 = 0, \ \varepsilon \neq 0)\) as follows:

\[
H(\mathbb{D}) = \{ Q = A + B i + C j + D k \mid A, B, C, D \in \mathbb{D}, i^2 = j^2 = k^2 = -1 = i j k \} \quad (1.5)
\]

In 2014, Nurkan and Güven [20] defined dual Fibonacci quaternions as follows:

\[
H(\mathbb{D}) = \{ \tilde{Q}_n = \tilde{F}_n + i \tilde{F}_{n+1} + j \tilde{F}_{n+2} + k \tilde{F}_{n+3} \mid \tilde{F}_n = F_n + \varepsilon F_{n+1}, \ \varepsilon^2 = 0, \ \varepsilon \neq 0 \}, \quad (1.6)
\]

where

\[
i^2 = j^2 = k^2 = i j k = -1, \quad i j = -j i = k, \quad j k = -k j = i, \quad k i = -i k = j
\]
$n \geq 1$ and $\tilde{Q}_n = Q_n + \varepsilon Q_{n+1}$. Essentially, these quaternions in equations (1.5) and (1.6) must be called dual coefficient quaternion and dual coefficient Fibonacci quaternions, respectively. For more details on dual quaternions, see [19]. It is clear that $H(\mathbb{D})$ and $H_D$ are different sets.

In 2016, Yüce and Torunbalcı Aydın [21] defined dual Fibonacci quaternions as follows:

$$H_D = \{ Q_n = F_n + i F_{n+1} + j F_{n+2} + k F_{n+3} \mid F_n, n \text{-th Fibonacci number} \},$$  \hspace{1cm} (1.7)

where

$$i^2 = j^2 = k^2 = i j k = 0, \quad i j = -j i = j k = -k j = k i = -i k = 0.$$

In 2016, Yüce and Torunbalcı Aydın [22] defined generalized dual Fibonacci quaternions as follows:

$$Q_D = \{ \mathbb{D}_n = H_n + i H_{n+1} + j H_{n+2} + k H_{n+3} \mid H_n, n \text{-th Generalized Fibonacci number} \},$$  \hspace{1cm} (1.8)

where

$$i^2 = j^2 = k^2 = i j k = 0, \quad i j = -j i = j k = -k j = k i = -i k = 0.$$

In 1971, Horadam studied on the Pell and Pell–Lucas sequences and he gave Cassini-like formula as follows [27]:

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n, \quad \text{(1.9)}$$

and Pell identities

$$\begin{cases}
P_r P_{n+1} + P_{r-1} P_n = P_{n+r}, \\
F_n (P_{n+1} + P_{n-1}) = P_{2n}, \\
P_{2n+1} + P_{2n} = 2P_{2n+1}^2 - 2P_n^2 - (-1)^n, \\
P_n^2 + P_{n+1}^2 = P_{2n+1}, \\
P_n^2 + P_{n+3}^2 = 5(P_{n+1}^2 + P_{n+2}^2), \\
P_{n+a} P_{n+b} - P_n P_{n+a+b} = (-1)^n P_n P_{n+a+b}, \\
P_{n-1} = (-1)^{n+1} P_n.
\end{cases} \quad \text{(1.10)}$$

In 1985, Horadam and Mohan [28] obtained Cassini-like formula as follows:

$$q_{n+1} q_{n-1} - q_n^2 = 8 (-1)^{n+1}. \quad \text{(1.11)}$$

First the idea to consider Pell quaternions it was suggested by Horadam in paper [12].

In 2017 (arXiv), Torunbalcı Aydin and Köklü [23] defined generalized Pell sequence as follows:

$$\begin{cases}
\mathbb{P}_0 = q, \ \mathbb{P}_1 = p, \ \mathbb{P}_2 = 2p + q, \ p q \in \mathbb{Z} \\
\mathbb{P}_n = 2P_{n-1} + \mathbb{P}_{n-2}, \ n \geq 2 \\
or \\
\mathbb{P}_n = (p - 2q)P_n + q P_{n+1} = p P_n + q P_{n-1}
\end{cases} \quad \text{(1.12)}$$

where $\mathbb{P}_n$ is the $n$-th generalized Pell number that defined in [23] as follows:

$$(\mathbb{P}_n) : q, \ p, \ 2p + q, \ 5p + 2q, \ 12p + 5q, \ 29p + 12q, \ldots, p P_n + q P_{n-1}, \ldots \quad \text{(1.13)}$$
In 2016, Torunbalcı Aydın and Yüce [24] defined dual Pell quaternions and dual Pell–Lucas quaternions as follows respectively:

\[ P_D = \{ D_P^n = P_n + i P_{n+1} + j P_{n+2} + k P_{n+3} \mid P_n \text{ n-th Pell number} \} \] \hspace{1cm} (1.14)

where

\[ i^2 = j^2 = k^2 = ij = -jk = k = -i k = 0 \]

and

\[ p_D = \{ D_p^n = q_n + i q_{n+1} + j q_{n+2} + k q_{n+3} \mid q_n \text{ n-th Pell–Lucas number} \} \] \hspace{1cm} (1.15)

\[ i^2 = j^2 = k^2 = ij = -jk = k = -i k = 0 . \]

Here, the Pell–Lucas sequence \((q_n)\) and \(q_n\) which is the \(n\)-th term of the dual Pell–Lucas quaternion sequence \((D_q^n)\) are defined by the following recurrence relations:

\[ (q_n) : 2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, \ldots, q_n, \ldots \]

\[ q_n = 2q_{n-1} + q_{n-2}, \quad n \geq 3, \]
\[ q_0 = 2, \quad q_1 = 2, \quad q_2 = 6. \] \hspace{1cm} (1.16)

In 2016, Çimen and İpek [25] worked on Pell quaternions and Pell–Lucas quaternions and defined as follows respectively:

\[ QP_n = \{ QP_n = P_n e_0 + P_{n+1} e_1 + P_{n+2} e_2 + P_{n+3} e_3 \mid P_n \text{ n-th Pell number} \} \] \hspace{1cm} (1.17)

and

\[ QPL_n = \{ QPL_n = q_n e_0 + q_{n+1} e_1 + q_{n+2} e_2 + q_{n+3} e_3 \mid q_n \text{ n-th Pell–Lucas number} \} \] \hspace{1cm} (1.18)

where

\[ e_0^2 = 1, \quad e_1^2 = e_2^2 = e_3^2 = -1, \]
\[ e_0 e_1 = e_1 e_0 = e_1, \quad e_0 e_2 = e_2 e_0 = e_2, \quad e_0 e_3 = e_3 e_0 = e_3, \]
\[ e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_3 e_1 = -e_1 e_3 = e_2. \]

In 2016, Anetta and Iwona [26] worked on the Pell quaternions and the Pell octanions. In this paper, we define the generalized dual Pell quaternions as follows:

\[ P_D = \{ P_D^n = P_n + i P_{n+1} + j P_{n+2} + k P_{n+3} \mid P_n \text{ n-th Gen.Pell number} \} \] \hspace{1cm} (1.19)

where

\[ i^2 = j^2 = k^2 = ij = -jk = k = -i k = 0. \]

Furthermore, we give Binet’s Formula and Cassini-like identities for the generalized dual Pell quaternions.
2 Generalized dual Pell quaternions

The generalized Pell sequence $\mathbb{P}_n$ is defined as

\[
\begin{align*}
\mathbb{P}_0 &= q, \mathbb{P}_1 = p, \mathbb{P}_2 = 2p + q, \ p, q \in \mathbb{Z} \\
\mathbb{P}_n &= 2\mathbb{P}_{n-1} + \mathbb{P}_{n-2}, \ n \geq 2 \\
\mathbb{P}_n &= (p - 2q)\mathbb{P}_n + q \mathbb{P}_{n+1} = p \mathbb{P}_n + q \mathbb{P}_{n-1}.
\end{align*}
\]

(2.1)

Here, $P_n$ is the $n$-th Pell number and $\mathbb{P}_n$ is the $n$-th generalized Pell number that defined in [23] as follows:

\[(\mathbb{P}_n) : q, p, 2p + q, 5p + 2q, 12p + 5q, 29p + 12q, \ldots, p \mathbb{P}_n + q \mathbb{P}_{n-1}, \ldots\]

We can define the generalized dual Pell quaternions by using generalized Pell numbers as follows

\[Q_D = \{D^P_n = \mathbb{P}_n + i \mathbb{P}_{n+1} + j \mathbb{P}_{n+2} + k \mathbb{P}_{n+3} | \mathbb{P}_n, n-th \ Gen. \ Pell \ number\}, \]

(2.2)

where

\[i^2 = j^2 = k^2 = i j k = 0, \ i j = -j i = j k = -k j = k i = -i k = 0.\]

The scaler and the vector part of $D^P_n$ which is the $n$-th term of the generalized dual Pell quaternion ($D^P_n$) are denoted by

\[S_{D^P_n} = \mathbb{P}_n \text{ and } V_{D^P_n} = i\mathbb{P}_{n+1} + j\mathbb{P}_{n+2} + k\mathbb{P}_{n+3}.\]

(2.3)

Thus, the generalized dual Pell quaternion $D^P_n$ is given by $D^P_n = S_{D^P_n} + V_{D^P_n}$. Let $D^{P_1}_n$ and $D^{P_2}_n$ be $n$-th terms of the generalized dual Pell quaternion sequences ($D^{P_1}_n$) and ($D^{P_2}_n$) such that

\[D^{P_1}_n = \mathbb{P}_n + i \mathbb{P}_{n+1} + j \mathbb{P}_{n+2} + k \mathbb{P}_{n+3}, \]

(2.4)

and

\[D^{P_2}_n = \mathbb{K}_n + i \mathbb{K}_{n+1} + j \mathbb{K}_{n+2} + k \mathbb{K}_{n+3}. \]

(2.5)

Then, the addition and subtraction of the generalized dual Pell quaternions is defined by

\[D^{P_1}_n \pm D^{P_2}_n = \left(\mathbb{P}_n + i \mathbb{P}_{n+1} + j \mathbb{P}_{n+2} + k \mathbb{P}_{n+3}\right) \pm \left(\mathbb{K}_n + i \mathbb{K}_{n+1} + j \mathbb{K}_{n+2} + k \mathbb{K}_{n+3}\right) \]

\(= (\mathbb{P}_n \pm \mathbb{K}_n) + i(\mathbb{P}_{n+1} \pm \mathbb{K}_{n+1}) + j(\mathbb{P}_{n+2} \pm \mathbb{K}_{n+2}) + k(\mathbb{P}_{n+3} \pm \mathbb{K}_{n+3}). \)

(2.6)

Multiplication of the generalized dual Pell quaternions is defined by

\[D^{P_1}_n \cdot D^{P_2}_n = \left(\mathbb{P}_n + i \mathbb{P}_{n+1} + j \mathbb{P}_{n+2} + k \mathbb{P}_{n+3}\right) \left(\mathbb{K}_n + i \mathbb{K}_{n+1} + j \mathbb{K}_{n+2} + k \mathbb{K}_{n+3}\right) \]

\(= (\mathbb{P}_n \mathbb{K}_n) + \mathbb{P}_n(i \mathbb{K}_{n+1} + j \mathbb{K}_{n+2} + k \mathbb{K}_{n+3}) + (i \mathbb{P}_{n+1} + j \mathbb{P}_{n+2} + k \mathbb{P}_{n+3}) \mathbb{K}_n. \)

(2.7)
or
\[ \mathbb{D}_P^1 \cdot \mathbb{D}_P^2 = S_{\mathbb{D}_P^1} S_{\mathbb{D}_P^2} + S_{\mathbb{D}_P^1} V_{\mathbb{D}_P^2} + S_{\mathbb{D}_P^2} V_{\mathbb{D}_P^1}. \] (2.8)

The conjugate of generalized dual Pell quaternion \( \mathbb{D}_P^P \) is denoted by \( \mathbb{D}_P^P \) and it is
\[ \mathbb{D}_P^P = P_n - i P_{n+1} - j P_{n+2} - k P_{n+3}. \] (2.9)

The norm of \( \mathbb{D}_P^P \) is defined as
\[ \| \mathbb{D}_P^P \|^2 = \mathbb{D}_P^P \mathbb{D}_P^P = (P_n)^2. \] (2.10)

Then, we give the following theorem using statements (2.1), (2.2) and the generalized Pell number in [23] as follows
\[ P_m P_{n+1} + P_{m-1} P_n = (2p - 2q)P_{m+n} - e_P P_{m+n} \] (2.11)

where \( e_P = p^2 - 2pq - q^2. \)

**Theorem 2.1.** Let \( P_n \) and \( \mathbb{D}_P^P \) be the \( n \) th terms of generalized Pell sequence \( (P_n) \) and the generalized dual Pell quaternion sequence \( (\mathbb{D}_P^P) \), respectively. In this case, for \( n \geq 1 \) we can give the following relations:
\[ \mathbb{D}_P^P + 2 \mathbb{D}_P^P_{n+1} = \mathbb{D}_P^P_{n+2} \] (2.12)
\[ (\mathbb{D}_P^P)^2 = 2 P_n \mathbb{D}_P^P - (P_n)^2 \] (2.13)
\[ \mathbb{D}_P^P - i \mathbb{D}_P^P_{n+1} - j \mathbb{D}_P^P_{n+2} - k \mathbb{D}_P^P_{n+3} = P_n \] (2.14)
\[ \mathbb{D}_P^P \mathbb{D}_P^P_m + \mathbb{D}_P^P_{n+1} \mathbb{D}_P^P_{m+1} = (2p - 2q)[2 \mathbb{D}_P^P_{n+m+1} - P_{n+m+1}] \] (2.15)

where \( D_{n+m+1}^P \) is the dual Pell quaternion [24].

**Proof.** (2.12): By the equations
\[ \mathbb{D}_P^P = P_n + i P_{n+1} + j P_{n+2} + k P_{n+3} \] (2.16)
and
\[ \mathbb{D}_P^P_{n+1} = P_{n+1} + i P_{n+2} + j P_{n+3} + k P_{n+4} \] (2.17)
we get,
\[ \mathbb{D}_P^P + 2 \mathbb{D}_P^P_{n+1} = (P_n + i P_{n+1} + j P_{n+2} + k P_{n+3}) \]
\[ + 2(P_{n+1} + i P_{n+2} + j P_{n+3} + k P_{n+4}) \]
\[ = (P_n + 2P_{n+1}) + i (P_{n+1} + 2P_{n+2}) + j (P_{n+2} + 2P_{n+3}) \]
\[ + k (P_{n+3} + 2P_{n+4}) \]
\[ = P_{n+2} + i P_{n+3} + j P_{n+4} + k P_{n+5} \]
\[ = \mathbb{D}_P^P_{n+2}. \] (2.18)
(2.13): 
\[
(D^P_n)^2 = (P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}) \\
= (P_n)^2 + 2i(P_nP_{n+1}) + 2j(P_nP_{n+2}) + 2k(P_nP_{n+3}) \\
= 2P_nD^P_n - (P_n)^2. 
\]

(2.14): By using (2.3) and conditions in the equation (2.2), we get
\[
D^P_n - iD^P_{n+1} - jD^P_{n+2} - kD^P_{n+3} = (P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}) \\
- i(P_{n+1} + iP_{n+2} + jP_{n+3} + kP_{n+4}) \\
- j(P_{n+2} + iP_{n+3} + jP_{n+4} + kP_{n+5}) \\
- k(P_{n+3} + iP_{n+4} + jP_{n+5} + kP_{n+6}) \\
= P_n.
\]

(2.15): By using (2.6) and (2.11)
\[
D^P_nD^P_m = P_nP_m + i(P_nP_{m+1} + P_{n+1}P_m) \\
+ j(P_nP_{m+2} + P_{n+2}P_m) + k(P_nP_{m+3} + P_{n+3}P_m). \\
D^P_{n+1}D^P_{m+1} = P_{n+1}P_{m+1} + i(P_{n+1}P_{m+2} + P_{n+2}P_{m+1}) \\
+ j(P_{n+1}P_{m+3} + P_{n+3}P_{m+1}) \\
+ k(P_{n+1}P_{m+4} + P_{n+4}P_{m+1}).
\]

Finally, adding equations (2.21) and (2.22) side by side, we obtain
\[
D^P_nD^P_m + D^P_{n+1}D^P_{m+1} = (P_nP_m + P_{n+1}P_{m+1}) \\
+ i[P_nP_{m+1} + P_{n+1}P_m + P_{n+1}P_{m+2} + P_{n+2}P_{m+1}] \\
+ j[P_nP_{m+2} + P_{n+2}P_m + P_{n+1}P_{m+3} + P_{n+3}P_{m+1}] \\
+ k[P_nP_{m+3} + P_{n+3}P_m + P_{n+1}P_{m+4} + P_{n+4}P_{m+1}] \\
= (2p - 2q)[P_{n+m+1} + 2iP_{n+m+2} + 2jP_{n+m+3} + 2kP_{n+m+4}] \\
- 2kP_{n+m+4}[P_{n+m+1} + 2iP_{n+m+2} + 2jP_{n+m+3} + 2kP_{n+m+4}] \\
= (2p - 2q)[2D^P_{n+m+1} - P_{n+m+1}] \\
- e[2D^P_{n+m+1} - P_{n+m+1}]
\]

where \(D^P_{n+m+1}\) is the dual Pell quaternion [24].

\[\square\]

**Theorem 2.2.** Let \(D^P_n\), \(D^P_n\) and \(D^q_n\) be \(n\)-th terms of the generalized dual Pell quaternion sequence \((D^P_n)\), the dual Pell quaternion sequence \((D^P_n)\) and the dual Pell–Lucas quaternion sequence \((D^P_n)\), respectively. The following relations are satisfied
\[
\begin{align*}
D^P_{n+1} + D^P_{n-1} &= pD^q_n + qD^q_{n-1}, \\
D^P_n + D^P_{n+1} &= \frac{p}{2}D^q_{n+1} + \frac{q}{2}D^q_n, \\
D^P_{n+1} - D^P_n &= \frac{p}{2}D^q_n + \frac{q}{2}D^q_{n-1}, \\
D^P_{n+1} - D^P_{n-1} &= 2[pD^P_n + qD^P_{n-1}], \\
D^P_{n+2} - D^P_{n-2} &= 2[pD^q_n + qD^q_{n-1}].
\end{align*}
\]
Proof. From equations (2.16), (2.17) and identities between the generalized Pell number $P_n$ [23],

$$
\begin{align*}
\mathbb{P}_n &= (p - 2q)P_n + q P_{n+1} = p P_n + q P_{n-1}, \\
\mathbb{P}_n + \mathbb{P}_{n+1} &= \frac{p}{2} q_{n+1} + \frac{q}{2} q_n, \\
\mathbb{P}_{n+1} - \mathbb{P}_n &= \frac{p}{2} q_n + \frac{q}{2} q_{n-1}, \\
\mathbb{P}_{n+1} + \mathbb{P}_{n-1} &= p q_n + q q_{n-1}, \\
\mathbb{P}_{n+1} - \mathbb{P}_{n-1} &= 2 (p P_n + q P_{n-1}), \\
\mathbb{P}_{n+2} - \mathbb{P}_{n-2} &= 2 (p q_n + q q_{n-1}).
\end{align*}
$$

(2.25)

also, from the relations of between Pell and Pell–Lucas numbers as follows:

$$
\begin{align*}
P_{n+1} + P_{n-1} &= q_n, \\
P_{n+1} - P_{n-1} &= 2 P_n, \\
P_n + P_{n+1} &= \frac{1}{2} q_{n+1}, \\
P_n - P_{n+1} &= 3 P_n, \\
P_{n+2} + P_{n-2} &= 6 P_n, \\
P_{n+2} - P_{n-2} &= 2 q_n.
\end{align*}
$$

(2.26)

it follows that

$$
\begin{align*}
\mathbb{D}P_{n+1} + \mathbb{D}P_{n-1} &= (\mathbb{P}_{n+1} + \mathbb{P}_{n-1}) + i (\mathbb{P}_{n+2} + \mathbb{P}_n) + j (\mathbb{P}_{n+3} + \mathbb{P}_{n+1}) + k (\mathbb{P}_{n+4} + \mathbb{P}_{n+2}) \\
&= [p (P_{n+1} + P_{n-1}) + q (P_n + P_{n-2})] \\
&+ i [p (P_{n+2} + P_n) + q (P_{n+1} + P_{n-1})] \\
&+ j [p (P_{n+3} + P_{n+1}) + q (P_{n+2} + P_n)] \\
&+ k [p (P_{n+4} + P_{n+2}) + q (P_{n+3} + P_{n+1})] \\
&= p (q_n + i q_{n+1} + j q_{n+2} + k q_{n+3}) \\
&+ q (q_{n-1} + i q_n + j q_{n+1} + k q_{n+2}) \\
&= \frac{p}{2} D_n^q + q D_{n-1}^q,
\end{align*}
$$

(2.27)

$$
\begin{align*}
\mathbb{D}P_n + \mathbb{D}P_{n+1} &= (\mathbb{P}_n + \mathbb{P}_{n+1}) + i (\mathbb{P}_{n+1} + \mathbb{P}_{n+2}) + j (\mathbb{P}_{n+2} + \mathbb{P}_{n+3}) + k (\mathbb{P}_{n+3} + \mathbb{P}_{n+4}) \\
&= [p (P_n + P_{n+1}) + q (P_{n-1} + P_n)] \\
&+ i [p (P_{n+1} + P_{n+2}) + q (P_n + P_{n+1})] \\
&+ j [p (P_{n+2} + P_{n+3}) + q (P_{n+1} + P_{n+2})] \\
&+ k [p (P_{n+3} + P_{n+4}) + q (P_{n+2} + P_{n+3})] \\
&= \frac{p}{2} (q_{n+1} + i q_{n+2} + j q_{n+3} + k q_{n+4}) \\
&+ \frac{q}{2} (q_n + i q_{n+1} + j q_{n+2} + k q_{n+3}) \\
&= \frac{p}{2} D_{n+1}^q + \frac{q}{2} D_n^q.
\end{align*}
$$
\[ \mathbb{D}^P_{n+1} - \mathbb{D}^P_n = \left( \mathbb{P}_{n+1} - \mathbb{P}_n \right) + i \left( \mathbb{P}_{n+2} - \mathbb{P}_{n+1} \right) + j \left( \mathbb{P}_{n+3} - \mathbb{P}_{n+2} \right) + k \left( \mathbb{P}_{n+4} - \mathbb{P}_{n+3} \right) \]

\[ = [p \left( P_{n+1} - P_n \right) + q \left( P_n - P_{n-1} \right)] + i [p \left( P_{n+2} - P_{n+1} \right) + q \left( P_{n+1} - P_n \right)] + j [p \left( P_{n+3} - P_{n+2} \right) + q \left( P_{n+2} - P_{n+1} \right)] + k [p \left( P_{n+4} - P_{n+3} \right) + q \left( P_{n+3} - P_{n+2} \right)] \]

\[ = \frac{p}{2} (q_n + i q_{n+1} + j q_{n+2} + k q_{n+3}) + \frac{q}{2} (q_{n-1} + i q_n + j q_{n+1} + k q_{n+2}) \]

\[ = \frac{p}{2} D^n_q + \frac{q}{2} D^n_p, \]

and

\[ \mathbb{D}^P_{n+1} - \mathbb{D}^P_{n-1} = \left( \mathbb{P}_{n+1} - \mathbb{P}_{n-1} \right) + i \left( \mathbb{P}_{n+2} - \mathbb{P}_n \right) + j \left( \mathbb{P}_{n+3} - \mathbb{P}_{n+1} \right) + k \left( \mathbb{P}_{n+4} - \mathbb{P}_{n+2} \right) \]

\[ = [p \left( P_{n+1} - P_{n-1} \right) + q \left( P_n - P_{n-2} \right)] + i [p \left( P_{n+2} - P_n \right) + q \left( P_{n+1} - P_{n-1} \right)] + j [p \left( P_{n+3} - P_{n+1} \right) + q \left( P_{n+2} - P_n \right)] + k [p \left( P_{n+4} - P_{n+2} \right) + q \left( P_{n+3} - P_{n+1} \right)] \]

\[ = 2 p \left( P_n + i P_{n+1} + j P_{n+2} + k P_{n+3} \right) + 2 q \left( P_{n-1} + i P_n + j P_{n+1} + k P_{n+2} \right) \]

\[ = 2 \left[ p D^n_p + q D^n_q \right], \]

\[ \text{Theorem 2.3.} \] Let \( \mathbb{D}^P_n \) be the \( n \)-th term of the generalized dual Pell quaternion sequence \( (\mathbb{D}^P_n) \). Then, we have the following relations between these quaternions:

\[ \mathbb{D}^P_n + \overline{\mathbb{D}^P_n} = 2 \mathbb{P}_n \]

\[ \mathbb{D}^P_n \overline{\mathbb{D}^P_n} + \mathbb{D}^P_{n-1} \overline{\mathbb{D}^P_{n-1}} = (\mathbb{P}_n)^2 + (\mathbb{P}_{n-1})^2 = (2p - 2q) \mathbb{P}_{2n-1} - e_p P_{2n-1} \]

\[ \mathbb{D}^P_n \overline{\mathbb{D}^P_{n+1}} + \mathbb{D}^P_{n+1} \overline{\mathbb{D}^P_n} = (\mathbb{P}_n)^2 + (\mathbb{P}_{n+1})^2 = (2p - 2q) \mathbb{P}_{2n+1} - e_p P_{2n+1} \]

\[ \mathbb{D}^P_{n+1} \overline{\mathbb{D}^P_{n+1}} - \mathbb{D}^P_{n-1} \overline{\mathbb{D}^P_{n-1}} = (\mathbb{P}_{n+1})^2 - (\mathbb{P}_{n-1})^2 = 2 [(2p - 2q) \mathbb{P}_{2n} - e_p P_{2n}] \]

\[ (\mathbb{D}^P_n)^2 + (\overline{\mathbb{D}^P_n})^2 = 2 \mathbb{D}^P_n \mathbb{P}_n + (\mathbb{P}_n)^2 + 2 \mathbb{D}^P_{n-1} \mathbb{P}_{n-1} - (\mathbb{P}_{n-1})^2 \]

\[ = (2p - 2q) \left[ 2 \mathbb{D}^P_{2n-1} - \mathbb{P}_{2n-1} \right] - e_p \left[ 2 D^p_{2n-1} - P_{2n-1} \right] \]

where \( D^p_{2n-1} \) is the dual Pell quaternion \([24]\).
Proof. (2.31): By using (2.9), we get
\[
\mathbb{D}^P_n + \overline{\mathbb{D}^P_n} = (\mathbb{P}_n + i \mathbb{P}_{n+1} + j \mathbb{P}_{n+2} + k \mathbb{P}_{n+3})
+ (\mathbb{P}_n - i \mathbb{P}_{n+1} - j \mathbb{P}_{n+2} - k \mathbb{P}_{n+3})
= 2 \mathbb{P}_n + i (\mathbb{P}_{n+1} - \mathbb{P}_{n+1}) + j (\mathbb{P}_{n+2} - \mathbb{P}_{n+2})
+ k (\mathbb{P}_{n+3} - \mathbb{P}_{n+3})
= 2 \mathbb{P}_n.
\]

(2.32): By using (2.9) and (2.10), we get
\[
\mathbb{D}^P_n \overline{\mathbb{D}^P_n} + \mathbb{D}^P_{n-1} \overline{\mathbb{D}^P_{n-1}} = (\mathbb{P}_n)^2 + (\mathbb{P}_{n-1})^2
= (2p - 2q) \mathbb{P}_{2n-1} - e_P P_{2n-1}
\]

(2.33): By using (2.9) and (2.10) and [23], we get
\[
\mathbb{D}^P_n \overline{\mathbb{D}^P_n} + \mathbb{D}^P_{n+1} \overline{\mathbb{D}^P_{n+1}} = (\mathbb{P}_n)^2 + (\mathbb{P}_{n+1})^2
= (2p - 2q) \mathbb{P}_{2n+1} - e_P P_{2n+1}
\]

(2.34): By using (2.9) and (2.10) and [23], we get
\[
\mathbb{D}^P_{n+1} \overline{\mathbb{D}^P_{n+1}} - \mathbb{D}^P_{n-1} \overline{\mathbb{D}^P_{n-1}} = (\mathbb{P}_{n+1})^2 - (\mathbb{P}_{n-1})^2
= (4p - 4q) \mathbb{P}_{2n} - 2 e_P P_{2n}
\]

(2.35): By using (2.10) and [23], we get
\[
(\mathbb{D}^P_n)^2 + (\mathbb{D}^P_{n-1})^2 = [2 \mathbb{D}^P_n \mathbb{P}_n - (\mathbb{P}_n)^2] + [2 \mathbb{D}^P_{n-1} \mathbb{P}_{n-1} - (\mathbb{P}_{n-1})^2]
= 2 \mathbb{D}^P_n \mathbb{P}_n + 2 \mathbb{D}^P_{n-1} \mathbb{P}_{n-1} - (\mathbb{P}_n)^2 + (\mathbb{P}_{n-1})^2
= (2p - 2q) [2 \mathbb{D}^P_{2n-1} - \mathbb{P}_{2n-1}] - e_P [2 D^n_{2n-1} - P_{2n-1}].
\]

where \( D^n_{2n-1} \) is the dual Pell quaternion [24]. □

**Theorem 2.4.** Let \( \mathbb{D}^P_n \) be the \( n \)–th term of the generalized dual Pell quaternion sequence \( (\mathbb{D}^P_n) \). Then, we have the following identities
\[
\sum_{s=1}^{n} \mathbb{D}^P_s = \frac{1}{4} [p D^n_{n+1} + q D^n_q] - \frac{p}{4} D^n_1 - \frac{q}{4} D^n_0, \quad (2.36)
\]
\[
\sum_{s=0}^{p} \mathbb{D}^P_{n+s} = \frac{p}{4} [D^n_{n+p+1} - D^n_q] + \frac{q}{4} [D^n_{n+p} - D^n_{n-1}], \quad (2.37)
\]
\[
\sum_{s=1}^{n} \mathbb{D}^P_{2s-1} = \frac{1}{2} [\mathbb{D}^P_{2n} - p D^n_0 - q D^n_{-1}], \quad (2.38)
\]
\[
\sum_{s=1}^{n} \mathbb{D}^P_{2s} = \frac{1}{2} [\mathbb{D}^P_{2n+1} - p D^n_1 - q D^n_0]. \quad (2.39)
\]

where \( D^n_P \) and \( D^n_q \) are the dual Pell quaternion and the dual Pell–Lucas quaternion respectively [24].
Proof. (2.36): Using \( \sum_{i=1}^{n} P_i = \frac{1}{2} (P_n + P_{n+1} - P_0 - P_1) \) [23], we get
\[
\sum_{s=1}^{n} D^P_s = \sum_{s=1}^{n} P_s + i \sum_{s=1}^{n} P_{s+1} + j \sum_{s=1}^{n} P_{s+2} + k \sum_{s=1}^{n} P_{s+3}
\]
\[
= \frac{1}{2} [(P_n + P_{n+1} - p - q) + i (P_{n+1} + P_{n+2} - 3p - q) \\
+ j (P_{n+2} + P_{n+3} - 7p - 3q) + k (P_{n+3} + P_{n+4} - 17p - 7q)]
\]
\[
= \frac{1}{2} (P_n + i P_{n+1} + j P_{n+2} + k P_{n+3})
\]
\[
+ \frac{1}{2} (P_{n+1} + i P_{n+2} + j P_{n+3} + k P_{n+4})
\]
\[
- \frac{p}{2} (1 + 3i + 7j + 17k) - \frac{q}{2} (1 + i + 3j + 7k)
\]
\[
= \frac{1}{2} [D^P_n + D^P_{n+1}] - \frac{p}{4} D^q_i - \frac{q}{4} D^q_0
\]
\[
= \frac{1}{4} [p D^q_{n+1} + q D^q_n] - \frac{p}{4} D^q_i - \frac{q}{4} D^q_0.
\]

(2.37): Hence, we can write
\[
\sum_{s=0}^{p} D^P_{n+s} = \sum_{s=0}^{p} P_{n+s} + i \sum_{s=0}^{p} P_{n+s+1} + j \sum_{s=0}^{p} P_{n+s+2} + k \sum_{s=0}^{p} P_{n+s+3}
\]
\[
= \frac{1}{2} [(P_{n+p} + P_{n+p+1} - P_1 - P_0) + i (P_{n+p+1} + P_{n+p+2} - P_2 - P_1) \\
+ j (P_{n+p+2} + P_{n+p+3} - P_3 - P_2) + k (P_{n+p+3} + P_{n+p+4} - P_4 - P_3)]
\]
\[
= \frac{1}{2} (P_{n+p} + i P_{n+p+1} + j P_{n+p+2} + k P_{n+p+3})
\]
\[
+ \frac{1}{2} (P_{n+p+1} + i P_{n+p+2} + j P_{n+p+3} + k P_{n+p+4})
\]
\[
- \frac{p}{2} (1 + 3i + 7j + 17k) - \frac{q}{2} (1 + i + 3j + 7k)
\]
\[
= \frac{1}{2} [D^P_{n+p} + D^P_{n+p+1}] - \frac{p}{4} D^q_i - \frac{q}{4} D^q_{n-1}
\]
\[
= \frac{p}{4} [D^q_{n+p+1} - D^q_n] + \frac{q}{4} [D^q_{n+p} - D^q_{n-1}].
\]

(2.38): Using \( \sum_{i=1}^{n} P_{2i-1} = \frac{1}{2} (P_{2n} - q) \) and \( \sum_{i=1}^{n} P_{2i} = \frac{1}{2} (P_{2n+1} - p) \) [23], we get
\[
\sum_{s=1}^{n} D^P_{2s-1} = \frac{1}{2} [(P_{2n} - q) + i (P_{2n+1} - p) + j (P_{2n+2} - q - 2p) \\
+ k (P_{2n+3} - 2q - 5p)]
\]
\[
= \frac{1}{2} [P_{2n} + i P_{2n+1} + j P_{2n+2} + k P_{2n+3}]
\]
\[
- \frac{1}{2} [q + ip + j(2p + q) + k(5p + 2q)]
\]
\[
= \frac{1}{2} [D^P_{2n} - p D^P_0 - q D^P_{-1}].
\]
(2.39): Using \( \sum_{i=1}^{n} P_{2i} = \frac{1}{2}(P_{2n+1} - p) \) [23], we obtain
\[
\sum_{s=1}^{n} D^P_{2s} = \frac{1}{2} \left[ (P_{2n+1} - p) + i (P_{2n+2} - 2p - q) + j (P_{2n+3} - 5p - 2q) + k (P_{2n+4} - 12p - 5q) \right] \\
= \frac{1}{2} \left[ P_{2n+1} + i P_{2n+2} + j P_{2n+3} + k P_{2n+4} \right] \\
- \frac{2}{3} \left[ 1 + 2i + 5j + 12k \right] - \frac{5}{3} \left[ 0 + i + 2j + 5k \right] \\
= \frac{1}{2} \left[ D^P_{2n+1} - p D^P_1 - q D^P_0 \right].
\]

**Theorem 2.5.** Let \( D^P_n \) and \( D^P_n \) be the \( n \)th terms of the generalized dual Pell quaternion sequence \( (D^P_n) \) and the dual Pell quaternion sequence \( (D^P_n) \), respectively. Then, we have
\[
D^P_n \overline{D^P_n} - D^P_n D^P_n = 2 \left[ P_n D^P_n - P_n \overline{D^P_n} \right] \tag{2.40}
\]
\[
D^P_n \overline{D^P_n} + D^P_n D^P_n = 2 P_n \overline{P_n} \tag{2.41}
\]
\[
D^P_n D^P_n - D^P_n \overline{D^P_n} = 2 \left[ P_n D^P_n + P_n D^P_n - 2 P_n \overline{P_n} \right] \tag{2.42}
\]

**Proof.** (2.40): By using (2.3) and (2.9), we get
\[
D^P_n \overline{D^P_n} - D^P_n D^P_n = (P_n + i P_{n+1} + j P_{n+2} + k P_{n+3}) \\
- (P_n - i P_{n+1} - j P_{n+2} - k P_{n+3}) \\
- (P_n - i P_{n+1} - j P_{n+2} - k P_{n+3}) \\
= (P_n P_n - P_n \overline{P_n}) + 2i (-P_n P_{n+1} + P_{n+1} P_n) \\
+ 2j (-P_n P_{n+2} + P_{n+2} P_n) \\
+ 2k (-P_n P_{n+3} + P_{n+3} \overline{P_n}) \\
= -2 P_n [P_n + i P_{n+1} + j P_{n+2} + k P_{n+3}] \\
+ 2 P_n [P_n + i P_{n+1} + j P_{n+2} + k P_{n+3}] \\
= 2 [P_n D^P_n - P_n \overline{D^P_n}].
\]

(2.41): By using (2.3) and (2.9), we get
\[
D^P_n \overline{D^P_n} + D^P_n D^P_n = (P_n + i P_{n+1} + j P_{n+2} + k P_{n+3}) \\
+ (P_n - i P_{n+1} - j P_{n+2} - k P_{n+3}) \\
+ (P_n - i P_{n+1} - j P_{n+2} - k P_{n+3}) \\
= (P_n P_n + P_n \overline{P_n}) \\
+ i (-P_n P_{n+1} + P_{n+1} P_n + P_n \overline{P_n} - P_{n+1} \overline{P_n}) \\
+ j (-P_n P_{n+2} + P_{n+2} P_n + P_n \overline{P_n} - P_{n+2} \overline{P_n}) \\
+ k (-P_n P_{n+3} + P_{n+3} P_n + P_n \overline{P_n} - P_{n+3} \overline{P_n}) \\
= 2 P_n \overline{P_n}.
\]
respectively. For Pell–Lucas sequence respectively, are as follows:

Proof. Respectively, where

\[ α = (p - qβ) + i [p (2 - β) + q] + j [p (5 - 2β) + q(2 - β)] + k [p (12 - 5β) + q (5 - 2β)], \quad α = 1 + √2, \]

\[ β = (qα - p) + i [p (α - 2) - q] + j [p (2α - 5) + q(α - 2)] + k [(p (5α - 12) + q (2α - 5)], \quad β = 1 - √2. \]

respectively.

\[ α = \frac{1}{α - β} \left( \hat{α} α^n - \hat{β} β^n \right) \] (2.43)

and

\[ α = (p - qβ) + i [p (2 - β) + q] + j [p (5 - 2β) + q(2 - β)] + k [p (12 - 5β) + q (5 - 2β)], \quad α = 1 + √2, \]

\[ β = (qα - p) + i [p (α - 2) - q] + j [p (2α - 5) + q(α - 2)] + k [(p (5α - 12) + q (2α - 5)], \quad β = 1 - √2. \]

respectively.

\[ α = \frac{1}{α - β} \left( \hat{α} α^n - \hat{β} β^n \right) \] (2.43)

and

\[ α = (p - qβ) + i [p (2 - β) + q] + j [p (5 - 2β) + q(2 - β)] + k [p (12 - 5β) + q (5 - 2β)], \quad α = 1 + √2, \]

\[ β = (qα - p) + i [p (α - 2) - q] + j [p (2α - 5) + q(α - 2)] + k [(p (5α - 12) + q (2α - 5)], \quad β = 1 - √2. \]

respectively.

Theorem 2.6 (Binet’s Formulas). Let \( D^P_n \) and \( D^q_n \) be \( n \)–th terms of the generalized dual Pell quaternion sequence \( \langle D^P_n \rangle \) and the generalized dual Pell–Lucas quaternion sequence \( \langle D^q_n \rangle \) respectively. For \( n \geq 1 \), the Binet’s formulas for these quaternions are as follows:

\[ D^P_n = \frac{1}{α - β} \left( \hat{α} α^n - \hat{β} β^n \right) \] (2.43)

and

\[ D^q_n = (\overline{α} α^n + \overline{β} β^n) \] (2.44)

respectively, where

\[ α = (p - qβ) + i [p (2 - β) + q] + j [p (5 - 2β) + q(2 - β)] + k [p (12 - 5β) + q (5 - 2β)], \quad α = 1 + √2, \]

\[ β = (qα - p) + i [p (α - 2) - q] + j [p (2α - 5) + q(α - 2)] + k [(p (5α - 12) + q (2α - 5)], \quad β = 1 - √2. \]

respectively.

Proof. The Binet’s formulas for Pell sequence, generalized Pell sequence and dual Pell quaternion sequence respectively, are as follows

\[ P_n = \frac{1}{2√2} (α^n - β^n) \], \( P_n = \frac{1}{2√2} (l α^n - m β^n) \) and \( D^P_n = \frac{1}{2√2} (α^n - β^n) \) [3],[23],[24].

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The roots of this equation are
\[ \alpha = 1 + \sqrt{2} \quad \text{and} \quad \beta = 1 - \sqrt{2}, \]
where \( \alpha + \beta = 2, \ \alpha - \beta = 2\sqrt{2}, \ \alpha\beta = -1. \)

Using recurrence relation and initial values \( (2.46) \) we get
\[ \begin{align*}
Dp_0 &= (q, \ p, \ 2p + q, \ 5p + 2q), \\
Dp_1 &= (p, \ 2p + q, \ 5p + 2q, \ 12p + 5q), \end{align*} \]
the Binet’s formula for \( D^p_n \) is
\[ D^p_n = A \alpha^n + B \beta^n = \frac{1}{2\sqrt{2}} \left[ \hat{\alpha} \alpha^n - \hat{\beta} \beta^n \right], \]
where \( A = \frac{D^p_1 - D^p_0}{\alpha - \beta} \) and \( B = \frac{\alpha D^p_0 - D^p_1}{\alpha - \beta} \) and
\[ \begin{align*}
\hat{\alpha} &= (p - q\beta) + i \left[ 2p(2 - \beta) + q \right] + j \left[ p(5 - 2\beta) + q(2 - \beta) \right] + k \left[ (12 - 5\beta) + q(5 - 2\beta) \right], \\
\hat{\beta} &= (q\alpha - p) + i \left[ p(\alpha - 2) - q \right] + j \left[ p(2\alpha - 5) + q(\alpha - 2) \right] + k \left[ p(5\alpha - 12) + q(2\alpha - 5) \right].
\end{align*} \]

Similarly, using recurrence relation \( D^q_n = 2D^q_{n+1} + D^q_n \), the Binet’s formula for generalized Pell–Lucas quaternion \( D^q_n \) is obtained as follows:
\[ D^q_n = (\hat{\alpha} \alpha^n + \hat{\beta} \beta^n) \] (2.45)
where initial values
\[ \begin{align*}
D^q_0 &= (2p - 2q, \ 2p + 2q, \ 6p + 2q, \ 14p + 6q), \\
D^q_1 &= (2p + 2q, \ 6p + 2q, \ 14p + 6q, \ 34p + 14q).
\end{align*} \]

**Theorem 2.7 (Cassini-like Identity).** Let \( D^p_n \) and \( D^q_n \) be \( n \)–th terms of the generalized dual Pell sequence \( (D^p_n) \) and the generalized dual Pell–Lucas sequence \( (D^p_n) \) respectively. For \( n \geq 1 \), the Cassini-like identity for \( D^p_n \) and \( D^p_n \) are as follows:
\[ D^p_{n-1} D^p_{n+1} - (D^p_n)^2 = (-1)^n e_P (1 + 2i + 6j + 14k) \] (2.46)
and
\[ D^q_{n-1} D^q_{n+1} - (D^q_n)^2 = 8(-1)^{n+1} e_q (1 + 2i + 6j + 14k) \] (2.47)
where
\[ e_P = e_q = p^2 - 2pq - q^2. \]

**Proof.** (2.46): By using (2.16) and (2.17) we get
\[ D^p_{n-1} D^p_{n+1} - (D^p_n)^2 = \]
\[ \begin{align*}
&= \left[ \left( (p_{n-1} + i p_n + j p_{n+1} + k p_{n+2}) \right) \right] \\
&\quad \left( (p_{n+1} + i p_{n+2} + j p_{n+3} + k p_{n+4}) \right) \\
&\quad - \left( (p_n + i p_{n+1} + j p_{n+2} + k p_{n+3}) \right)^2 \\
&= \left[ (p_{n-1} p_{n+1} - (p_n)^2) \right] \\
&\quad + i \left[ p_{n-1} p_{n+2} + p_n p_{n+1} - 2 p_n p_{n+1} \right] \\
&\quad + j \left[ p_{n-1} p_{n+3} - 2 p_n p_{n+2} + (p_{n+1})^2 \right] \\
&\quad + k \left[ p_{n-1} p_{n+4} + p_n p_{n+3} - 2 p_{n+1} p_{n+2} \right] \\
&= (-1)^n e_P (1 + 2i + 6j + 14k).
\end{align*} \]
where we use identity of the Pell number $P_m P_{n+1} - P_{m+1} P_n = (-1)^n P_{m-n}$ and identities of the generalized Pell numbers as follows:

\begin{align*}
  P_{n+1} P_{n-1} - (P_n)^2 &= (-1)^n e_P, \quad (2.48) \\
  P_{n+2} P_{n-1} - P_n P_{n+1} &= 2(-1)^n e_P, \quad (2.49) \\
  P_{n+3} P_{n-1} + P_{n+1} P_{n+1} - 2 P_n P_{n+2} &= 6 (-1)^n e_P, \quad (2.50) \\
  P_{n+4} P_{n-1} + P_{n+2} P_{n+1} - 2 P_n P_{n+3} &= 14 (-1)^n e_P, \quad (2.51)
\end{align*}

Let the generalized Pell–Lucas sequence $(q_n)$ be defined as follows:

\begin{align*}
  q_0 &= 2p - 2q, \quad q_1 = 2p + 2q, \quad q_2 = 6p + 2q, \quad p, q \in \mathbb{Z} \\
  q_n &= 2q_{n-1} + q_{n-2}, \quad n \geq 2 \\
  \text{or} \quad q_n &= (p - 2q) q_n + q q_{n+1} = p q_n + q q_{n-1}. \quad (2.52)
\end{align*}

Here, $q_n$ is the $n$-th generalized Pell–Lucas number that defined as follows:

\begin{align*}
  (q_n) : 2p - 2q, \quad 2p + 2q, \quad 6p + 2q, \quad 14p + 6q, \quad 34p + 14q, \ldots, p q_n + q q_{n-1}, \ldots \quad (2.53)
\end{align*}

and let the generalized dual Pell–Lucas quaternion be defined as follows:

\begin{align*}
  \mathbb{D}^q_n = q_n + i q_{n+1} + j q_{n+2} + k q_{n+3} \mid q_n, \text{ n-th gen. Pell–Lucas number} \quad (2.54)
\end{align*}

where

\begin{align*}
  i^2 = j^2 = k^2 = ij = jk = ki = -jk = k = 0.
\end{align*}

(2.47): By using (2.53) and (2.54) we get

\begin{align*}
  \mathbb{D}^q_{n-1} \mathbb{D}^q_{n+1} - (\mathbb{D}^q_n)^2 &= (q_{n-1} + i q_n + j q_{n+1} + k q_{n+2}) \\
  &\quad (q_{n+1} + i q_{n+2} + j q_{n+3} + k q_{n+4}) \\
  &\quad - (q_n + i q_{n+1} + j q_{n+2} + k q_{n+3})^2 \\
  &= \left[ q_{n-1} q_{n+1} - (q_n)^2 \right] \\
  &\quad + i \left[ q_{n-1} q_{n+2} + q_n q_{n+1} - 2 q_n q_{n+1} \right] \\
  &\quad + j \left[ q_{n-1} q_{n+3} - 2 q_n q_{n+2} + (q_{n+1})^2 \right] \\
  &\quad + k \left[ q_{n-1} q_{n+4} + q_{n+1} q_{n+2} - 2 q_n q_{n+3} \right] \\
  &= 8 (-1)^{n+1} e_q (1 + 2i + 6j + 14k).
\end{align*}

where we use identity of the Pell–Lucas number $q_{n-1} q_{n+1} - q_n q_n = 8 (-1)^{n+1}$ and identities of the generalized Pell–Lucas numbers as follows:

\begin{align*}
  q_{n+1} q_{n-1} - (q_n)^2 &= 8 (-1)^{n+1} e_q, \quad (2.55) \\
  q_{n+2} q_{n-1} - q_n q_{n+1} &= 16(-1)^{n+1} e_q, \quad (2.56)
\end{align*}
Example 1. Let $D^P_1, D^P_2, D^P_3$ and $D^P_4$ be the generalized dual Pell quaternions such that

\[
\begin{align*}
D^P_1 &= p + i(2p + q) + j(5p + 2q) + k(12p + 5q) \\
D^P_2 &= (2p + q) + i(5p + 2q) + j(12p + 5q) + k(29p + 12q) \\
D^P_3 &= (5p + 2q) + i(12p + 5q) + j(29p + 12q) + k(70p + 29q) \\
D^P_4 &= (12p + 5q) + i(29p + 12q) + j(70p + 29q) + k(169p + 70q).
\end{align*}
\]

In this case,

\[
\begin{align*}
D^P_1 D^P_3 - (D^P_2)^2 &= \left[p + i(2p + q) + j(5p + 2q) + k(12p + 5q)\right] \\
&\left[(5p + 2q) + i(12p + 5q) + j(29p + 12q) + k(70p + 29q)\right] \\
&\left[-(2p + q) + i(5p + 2q) + j(12p + 5q) + k(29p + 12q)\right]^2 \\
&= (p^2 - 2pq - q^2) + i(2p^2 - 4pq - 2q^2) \\
&+ j(6p^2 - 12pq - 6q^2) + k(14p^2 - 28pq - 14q^2) \\
&= (p^2 - 2pq - q^2)(1 + 2i + 6j + 14k) \\
&= (-1)^2 e_P (1 + 2i + 6j + 14k) \\
\end{align*}
\]

and

\[
\begin{align*}
D^P_2 D^P_4 - (D^P_3)^2 &= \left[(2p + q) + i(5p + 2q) + j(12p + 5q) + k(29p + 12q)\right] \\
&\left[(12p + 5q) + i(29p + 12q) + j(70p + 29q) + k(169p + 70q)\right] \\
&\left[-(5p + 2q) + i(12p + 5q) + j(29p + 12q) + k(70p + 29q)\right]^2 \\
&= (-p^2 + 2pq + q^2) + i(-2p^2 + 4pq + 2q^2) \\
&+ j(-6p^2 + 12pq + 6q^2) + k(-14p^2 + 28pq + 14q^2) \\
&= -(p^2 - 2pq - q^2)(1 + 2i + 6j + 14k) \\
&= (-1)^3 e_P (1 + 2i + 6j + 14k).
\end{align*}
\]
Example 2. Let $\mathbb{D}^{q_1}, \mathbb{D}^{q_2}, \mathbb{D}^{q_3}$ and $\mathbb{D}^{q_4}$ be the generalized dual Pell–Lucas quaternions such that

$$
\begin{align*}
\mathbb{D}^{q_1} &= (2p + 2q) + i(6p + 2q) + j(14p + 6q) + k(34p + 14q) \\
\mathbb{D}^{q_2} &= (6p + 2q) + i(14p + 6q) + j(34p + 14q) + k(82p + 34q) \\
\mathbb{D}^{q_3} &= (14p + 6q) + i(34p + 14q) + j(82p + 34q) + k(198p + 82q) \\
\mathbb{D}^{q_4} &= (34p + 14q) + i(82p + 34q) + j(198p + 82q) + k(478p + 198q).
\end{align*}
$$

In this case,

$$
\begin{align*}
\mathbb{D}^{q_1}\mathbb{D}^{q_3} - (\mathbb{D}^{q_2})^2 &= [(2p + 2q) + i(6p + 2q) + j(14p + 6q) + k(34p + 14q)] \\
&\quad \times [(14p + 6q) + i(34p + 14q) + j(82p + 34q) + k(198p + 82q)] \\
&\quad - [(6p + 2q) + i(14p + 6q) + j(34p + 14q) + k(82p + 34q)]^2 \\
&= -8(p^2 - 16pq - 8q^2) - i(16p^2 - 32pq - 16q^2) \\
&\quad \times -j(48p^2 - 160pq - 48q^2) - k(112p^2 - 224pq - 112q^2) \\
&= -8(p^2 - 2pq - q^2)(1 + 2i + 6j + 14k) \\
&= 8(-1)^3e_q(1 + 2i + 6j + 14k)
\end{align*}
$$

and

$$
\begin{align*}
\mathbb{D}^{q_2}\mathbb{D}^{q_4} - (\mathbb{D}^{q_3})^2 &= [(6p + 2q) + i(14p + 6q) + j(34p + 14q) + k(82p + 34q)] \\
&\quad \times [(34p + 14q) + i(82p + 34q) + j(198p + 82q) + k(478p + 198q)] \\
&\quad - [(14p + 6q) + i(34p + 14q) + j(82p + 34q) + k(198p + 82q)]^2 \\
&= 8(p^2 - 2pq - q^2) + 16i(p^2 - 2pq - q^2) \\
&\quad \times +48j(p^2 - 2pq - q^2) + 112k(p^2 - 2pq - q^2) \\
&= 8(p^2 - 2pq - q^2)(1 + 2i + 6j + 14k) \\
&= 8(-1)^4e_q(1 + 2i + 6j + 14k).
\end{align*}
$$

3 Conclusion

The generalized dual Pell quaternions is given by

$$
\mathbb{D}^{P_n} = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3},
$$

where $P_n$ is the $n$-th generalized Pell number and $i, j, k$ are quaternionic units which satisfy the equalities

$$
i^2 = j^2 = k^2 = 0, \quad i j = -j i = j k = -k j = k i = -i k = 0.$$

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The generalized dual Pell–Lucas quaternions is given by

\[ \mathbb{D}^q_n = q_n + i q_{n+1} + j q_{n+2} + k q_{n+3}, \]  

(3.2)

where \( q_n \) is the \( n \)-th generalized Pell–Lucas number and \( i, j, k \) are quaternionic units which satisfy the equalities

\[ i^2 = j^2 = k^2 = 0, \quad i j = -j i = j k = -k j = k i = -i k = 0. \]

Also, from the generalized dual Pell quaternions and the generalized dual Pell–Lucas quaternions for \( p = 1, q = 0 \), we obtain results of the dual Pell quaternions and the dual Pell–Lucas quaternions given by Torunbalcı Aydın and Yüce [24] as a special case.

References


