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Two applications of the Hadamard integral inequality

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Abstract: As applications of the Hadamard integral inequality, we offer two inequalities for trigonometric, resp. hyperbolic functions. One of results gives a new proof of the Iyengar–Madhava Rao–Nanjundiah inequality for $\frac{\sin x}{x}$.
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1 Introduction

The famous Hadamard integral inequality states that for any continuous, convex function $f : [a, b] \to \mathbb{R}$ ($a < b$ real numbers), one has the inequality

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \geq f\left(\frac{a+b}{2}\right). \tag{1}$$

The inequality in (1) is strict, if f is a strictly convex function.

In 1945, K. S. K. Iyengar, B. S. Madhava Rao and T. S. Nanjundiah [1], in a little known paper, have shown that for any $x \in \left(0, \frac{\pi}{2}\right)$ one has

$$\frac{\sin x}{x} > \cos\left(\frac{x}{\sqrt{3}}\right). \tag{2}$$

We note, that, inequality (2) refines the better known, and also famous inequality by Adamović–Mitrinović [2]:

$$\frac{\sin x}{x} > \sqrt[3]{\cos x}. \tag{3}$$

This follows by the inequality

$$\cos\left(\frac{x}{\sqrt{3}}\right) > \sqrt[3]{\cos x}. \tag{4}$$

In what follows, we will offer a new proof to inequality (2), as well as (4), and offer also the hyperbolic version of (2), namely: for any $x > 0$ one has

$$\frac{\sinh x}{x} > \cosh\left(\frac{x}{\sqrt{3}}\right). \tag{5}$$

Our method will be based on Hadamard's inequality (1).

2 Main results

Theorem 2.1. *Suppose that $\lambda > 1, t > 0$ and $\lambda t \in \left(0, \frac{\pi}{2}\right)$. Then*

$$\cos t - \cos(\lambda t) < \frac{\lambda^2 - 1}{2} t \sin t. \tag{6}$$

Proof. Apply the Hadamard inequality (1) to $f(x) = -\sin x, a = t, b = \lambda t$. Then we get the relation

$$\frac{\cos t - \cos \lambda t}{t(\lambda - 1)} < \sin \frac{t(\lambda + 1)}{2}. \tag{7}$$

Now, it is well-known (see e.g. [3]) that the function $u \to \dfrac{\sin u}{u}$ is strictly decreasing in $\left(0, \frac{\pi}{2}\right)$. This implies that for any $a > 1, t > 0$ one has $\dfrac{\sin ta}{ta} < \dfrac{\sin t}{t}$, implying

$$\sin ta < a \sin t. \tag{8}$$

Particularly, for $a = \dfrac{\lambda + 1}{2} > 1$ we get that $\sin \dfrac{t(\lambda + 1)}{2} < \dfrac{\lambda + 1}{2} \sin t$. Combining this with (7), relation (6) follows. □

Particularly, for $\lambda = \sqrt{3}$ we get by (6):

$$\cos t - \cos(\sqrt{3}t) < t \sin t \tag{9}$$

for $\sqrt{3}t \in \left(0, \frac{\pi}{2}\right)$.

The hyperbolic variant of (6) is contained in the following

Theorem 2.2. *Let $\lambda > 1$ and $t > 0$. Then*

$$\cosh(\lambda t) - \cosh t > \frac{\lambda^2 - 1}{2} t \sinh t. \tag{10}$$

Proof. Apply (1) to $f(x) = \sinh x, a = t, b = \lambda t$. Remarking that the function $u \to \dfrac{\sinh u}{u}$ is strictly increasing for $u > 0$, we get (10) as in the proof of (6). □

Particularly, for $\lambda = \sqrt{3}$ we get by (10):

$$\cosh(\sqrt{3}t) - \cosh t > t \sinh t, \tag{11}$$

for any $t > 0$.

As an application of (9), we get:

Theorem 2.3.

$$\frac{\sin x}{x} > \cos\left(\frac{x}{\sqrt{3}}\right) > \sqrt[3]{\cos x} \tag{12}$$

for $x \in \left(0, \frac{\pi}{2}\right)$.

Proof. Let $x = \sqrt{3}t \in \left(0, \frac{\pi}{2}\right)$, and introduce $g(t) = \sin(\sqrt{3}t) - \sqrt{3}t \cos t$. One has immediately $g'(t) = \sqrt{3}(\cos\sqrt{3}t - \cos t + t \sin t) > 0$ by (9). This gives $g(t) > g(0) = 0$, and the first inequality of (12) follows.

For the second inequality, put $h(t) = \cos^3 t - \cos(\sqrt{3}t)$.

As $h'(t) = \sqrt{3}[\sin(\sqrt{3}t) - \sqrt{3}\sin t \cos^2 t] > \sqrt{3}\cos t(t - \sin t \cos t) > 0$ by $\sin t \cos t < \sin t < t$. We have used the inequality $\sin(\sqrt{3}t) > \sqrt{3}t \cos t$, which follows by the first part of (12). By letting $t = \dfrac{x}{\sqrt{3}}$, the second inequality of (12) follows. □

Theorem 2.4. *For any* $x > 0$ *one has*

$$\frac{\sinh x}{x} > \cosh \frac{x}{\sqrt{3}}. \tag{13}$$

Proof. Put $x = \sqrt{3}t$ and consider $k(t) = \sinh(\sqrt{3}t) - \sqrt{3}t \cosh t$. It is immediate that $k'(t) = \sqrt{3}[\cosh(\sqrt{3}t) - \cosh t - t \sinh t] > 0$ by (11). This implies $k(t) > k(0) = 0$, and inequality (13) is established. □

Remark 1. *It can be proved by other methods that, inequality* (13) *refines the famous Lazarević inequality (see [2])*

$$\frac{\sinh x}{x} > \sqrt[3]{\cosh x}, x > 0. \tag{14}$$

It should be shown that

$$\cosh \frac{x}{\sqrt{3}} > \sqrt[3]{\cosh x} \tag{15}$$

for $x > 0$. *Indeed, put* $x = \sqrt{3}t$. *As* (15) *is equivalent with* $\cosh^3 t > \cosh(\sqrt{3}t)$, *remark that by the identity* $\cosh^3 t = \dfrac{\cosh(3t) + 3\cosh t}{4}$, *we have to show that*

$$\cosh(3t) + 3\cosh t > 4\cosh(\sqrt{3}t). \tag{16}$$

By using the series expansion $\cosh x = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{24} + \dfrac{x^6}{720} + \cdots$, *it is immediate that the left side of* (16) *is* $t^2 \cdot \dfrac{12}{2} + t^4\left(\dfrac{3^4+3}{24}\right) + t^6\left(\dfrac{3^6+3}{720}\right) + \cdots$, *while right side is* $t^2 \cdot \dfrac{12}{2} + t^4 \dfrac{4 \cdot 3^2}{24} + t^6\left(\dfrac{4 \cdot 3^3}{720}\right) + \cdots$, *so it is sufficient to prove that* $3^4 + 3 > 4 \cdot 3^2$, $3^6 + 3 > 4 \cdot 3^3, \ldots$ *and generally*

$$3^{2n+2} + 3 > 4 \cdot 3^{n+1}, n \geq 1, \tag{17}$$

which easily follows by mathematical induction.

References

[1] Iyengar, K. S. K., Madhava Rao, B. S., & Nanjundiah, T. S. (1945) Some trigonometrical inequalities, *Half-Yearly J. Mysore Univ. Sect. B., NS*, 6, 1–12.

[2] Mitrinović, D. S. (1970) *Analytic inequalities*, Springer–Verlag, Berlin.

[3] Sándor, J. (2005) On the concavity of $\frac{\sin x}{x}$, *Octogon Math. Mag.*, 13(1), 406–407.