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On limits and formulae where functions of slow increase appear

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In memory of my sister Fedra Marina Jakimczuk (1970–2010)

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Abstract: Let A_n be a strictly increasing sequence of positive integers such that

$$A_n \sim n^s f(n),$$

where $f(x)$ is a function of slow increase and s is a positive real number. In this article we obtain some limits and asymptotic formulae where appear functions of slow increase. As example, we apply the obtained results to the sequence of numbers with exactly k prime factors in their prime factorization, in particular to the sequence of prime numbers ($k = 1$).

Keywords: Functions of slow increase, Integer sequences, The e number, Limits, Numbers with a fixed number of primes in their prime factorization.
AMS Classification: 11A99, 11B99.

1 Main results

Definition 1.1. *Let $f(x)$ be a function defined on the interval $[a, \infty)$ such that $f(x) > 0$, $\lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$. The function $f(x)$ is of slow increase if and only if the following condition holds*

$$\lim_{x\to\infty} \frac{f'(x)}{\frac{f(x)}{x}} = 0 \tag{1}$$

Typical functions of slow increase are $f(x) = \log x$, $f(x) = \log^2 x$, $f(x) = \log\log x$, etc.

Let A_n be a strictly increasing sequence of positive integers such that

$$A_n \sim n^s f(n),$$

where $f(x)$ is a function of slow increase. For example, the sequence $c_{n,k}$ of numbers that have exactly k prime factors in their prime factorization. In particular, if $k = 1$ then $c_{n,1} = p_n$ is the sequence of prime numbers. Since (see [2]) we have

$$c_{n,k} \sim n\frac{(k-1)!\log n}{(\log\log n)^{k-1}}$$

and the function

$$f(x) = \frac{(k-1)!\log x}{(\log\log x)^{k-1}}$$

is of slow increase. Here $s = 1$.

In [2], we prove the limit

$$\lim_{n\to\infty} \frac{(A_1 A_2 \cdots A_n)^{\frac{1}{n}}}{A_n} = \frac{1}{e^s} \tag{2}$$

For example

$$\lim_{n\to\infty} \frac{(c_{1,k} c_{2,k} \cdots c_{n,k})^{\frac{1}{n}}}{c_{n,k}} = \frac{1}{e}$$

The case $k = 1$, namely

$$\lim_{n\to\infty} \frac{(p_1 p_2 \cdots p_n)^{\frac{1}{n}}}{p_n} = \frac{1}{e}$$

has been studied in [1, 4, 5].

In the following theorem we generalize limit (2). We have the following theorem.

Theorem 1.2. *Let A_n be a strictly increasing sequence of positive integers such that*

$$A_n \sim n^s f(n), \tag{3}$$

where $f(x)$ is a function of slow increase and s is a positive real number.

Let k be a fixed but arbitrary nonnegative integer. We prove the limit

$$\lim_{n\to\infty} \frac{\left(A_1^{(1^k)} A_2^{(2^k)} \cdots A_n^{(n^k)}\right)^{\frac{k+1}{n^{k+1}}}}{A_n} = \frac{1}{e^{\frac{s}{k+1}}}. \tag{4}$$

Proof. If $k = 0$, then the theorem is true (see (2)). Suppose that k is a positive integer. Note that without loss of generality we can suppose in Definition 1.1 that $a = 1$ and $f(1) \geq 1$. We have (see (3))

$$A_i \sim i^s f(i) \qquad (i \geq 1)$$

Therefore

$$\log A_i = s\log i + \log f(i) + g(i) \qquad (i \geq 1), \tag{5}$$

where

$$g(i) \to 0. \tag{6}$$

Now, we have (see (5))

$$\begin{aligned} & \log\left(A_1^{(1^k)} A_2^{(2^k)} \cdots A_n^{(n^k)}\right) \\ = & \sum_{i=1}^{n} i^k \log A_i = \sum_{i=1}^{n} \left(s i^k \log i + i^k \log f(i) + g(i) i^k\right) \\ = & s \sum_{i=1}^{n} i^k \log i + \sum_{i=1}^{n} i^k \log f(i) + \sum_{i=1}^{n} g(i) i^k. \end{aligned} \tag{7}$$

The function $x^k \log x$ is nonnegative and strictly increasing on the interval $[1, \infty)$. Consequently

$$s \sum_{i=1}^{n} i^k \log i = s \int_1^n x^k \log x \, dx + O(n^k \log n) \tag{8}$$

Note that the sum in the left side is a sum of rectangles of basis 1 and height $i^k \log i$.

The function $x^k \log f(x)$ is nonnegative and strictly increasing on the interval $[1, \infty)$. Consequently

$$\sum_{i=1}^{n} i^k \log f(i) = \int_1^n x^k \log f(x) \, dx + O(n^k \log f(n)). \tag{9}$$

Note that the sum in the left side is a sum of rectangles of basis 1 and height $i^k \log f(i)$.

We have (use integration by parts)

$$s \int_1^n x^k \log x \, dx = s \frac{n^{k+1}}{k+1} \log n - \frac{s}{(k+1)^2} n^{k+1} + O(1). \tag{10}$$

On the other hand, we have (use integration by parts)

$$\int_1^n x^k \log f(x) \, dx = \frac{n^{k+1}}{k+1} \log f(n) + O(1) - \frac{1}{k+1} \int_1^n \frac{x^{k+1} f'(x)}{f(x)} \, dx. \tag{11}$$

If the integral

$$\int_1^x \frac{t^{k+1} f'(t)}{f(t)} \, dt \tag{12}$$

is divergent then we have (L'Hospital's rule and Equation (1))

$$\lim_{x \to \infty} \frac{\int_1^x \frac{t^{k+1} f'(t)}{f(t)} \, dt}{x^{k+1}} = \lim_{x \to \infty} \frac{\frac{x^{k+1} f'(x)}{f(x)}}{(k+1) x^k} = 0 \tag{13}$$

Therefore, (13) gives

$$\lim_{n \to \infty} \frac{\int_1^n \frac{x^{k+1} f'(x)}{f(x)} \, dx}{n^{k+1}} = 0$$

That is

$$\int_1^n \frac{x^{k+1} f'(x)}{f(x)}\, dx = o\left(n^{k+1}\right). \tag{14}$$

On the other hand, if the integral (12) is convergent, clearly (14) also holds.
Equations (11) and (14) give

$$\int_1^n x^k \log f(x)\, dx = \frac{n^{k+1}}{k+1} \log f(n) + o\left(n^{k+1}\right). \tag{15}$$

Note that (L'Hospital rule and (1)) we have

$$\lim_{x\to\infty} \frac{\log f(x)}{x} = \lim_{x\to\infty} \frac{f'(x)}{f(x)} = \lim_{x\to\infty} \frac{x f'(x)}{f(x)} \frac{1}{x} = 0.0 = 0. \tag{16}$$

Equations (8) and (10) give

$$s \sum_{i=1}^{n} i^k \log i = s \frac{n^{k+1}}{k+1} \log n - \frac{s}{(k+1)^2} n^{k+1} + o(n^{k+1}). \tag{17}$$

Equations (9), (15) and limit (16) give

$$\sum_{i=1}^{n} i^k \log f(i) = \frac{n^{k+1}}{k+1} \log f(n) + o(n^{k+1}). \tag{18}$$

Given $\epsilon > 0$, there exist n_0 such that if $n \geq n_0$ we have $|g(i)| < \epsilon$ (see (6)). Therefore,

$$\left|\sum_{i=1}^{n} g(i) i^k\right| \leq \sum_{i=1}^{n} |g(i)|\, i^k \leq \sum_{i=1}^{n_0-1} |g(i)|\, i^k + \epsilon \sum_{i=n_0}^{n} i^k \leq \sum_{i=1}^{n_0-1} |g(i)|\, i^k + \epsilon \sum_{i=1}^{n} i^k. \tag{19}$$

Now

$$\sum_{i=1}^{n} i^k = \int_1^n x^k\, dx + O(n^k) = \frac{n^{k+1}}{k+1} + o(n^{k+1}). \tag{20}$$

Therefore (see (19) and (20)) from a certain value of n we have

$$\left|\frac{\sum_{i=1}^{n} g(i) i^k}{n^{k+1}}\right| \leq \frac{\sum_{i=1}^{n_0-1} |g(i)|\, i^k}{n^{k+1}} + \epsilon \frac{\sum_{i=1}^{n} i^k}{n^{k+1}} \leq \epsilon.$$

That is

$$\sum_{i=1}^{n} g(i) i^k = o(n^{k+1}). \tag{21}$$

Substituting (17), (18) and (21) into (7) we obtain

$$\begin{aligned} \log\left(A_1^{(1^k)} A_2^{(2^k)} \cdots A_n^{(n^k)}\right) &= s \frac{n^{k+1}}{k+1} \log n + \frac{n^{k+1}}{k+1} \log f(n) \\ - \quad & s \frac{n^{k+1}}{(k+1)^2} + o(n^{k+1}). \end{aligned} \tag{22}$$

Therefore (see (22), (5) and (6)) we have

$$\log\left(\frac{\left(A_1^{(1^k)}A_2^{(2^k)}\cdots A_n^{(n^k)}\right)^{\frac{k+1}{n^{k+1}}}}{A_n}\right)=\frac{k+1}{n^{k+1}}\log\left(A_1^{(1^k)}A_2^{(2^k)}\cdots A_n^{(n^k)}\right)$$

$$-\quad \log A_n=-\frac{s}{k+1}+o(1). \tag{23}$$

That is, limit (4). □

For example, we have

$$\lim_{n\to\infty}\frac{\left(c_{1,k}^{(1^k)}c_{2,k}^{(2^k)}\cdots c_{n,k}^{(n^k)}\right)^{\frac{k+1}{n^{k+1}}}}{c_{n,k}}=\frac{1}{e^{\frac{1}{k+1}}}.$$

In particular

$$\lim_{n\to\infty}\frac{\left(p_1^{(1^k)}p_2^{(2^k)}\cdots p_n^{(n^k)}\right)^{\frac{k+1}{n^{k+1}}}}{p_n}=\frac{1}{e^{\frac{1}{k+1}}}.$$

This limit was proved in a former article, see [3].

Let A_n be a strictly increasing sequence of positive integers such that

$$A_n\sim n^s f(n),$$

where $f(x)$ is a function of slow increase. In a previous article [2], we prove the asymptotic formula

$$\sum_{i=1}^{n}\log A_i=sn\log n+n\log f(n)-sn+o(n). \tag{24}$$

Theorem 1.3. *Let A_n be a strictly increasing sequence of positive integers such that*

$$A_n\sim n^s f(n), \tag{25}$$

where $f(x)$ is a function of slow increase and s is a positive real number.

The following asymptotic formula holds

$$\sum_{i=1}^{n}\frac{\log A_i}{i}=\frac{s}{2}\log^2 n+\int_b^n\frac{\log f(x)}{x}\,dx+o(\log n) \tag{26}$$

Proof. Let $b>a$ a positive integer such that $f(b)>1$. Equation (25) gives

$$\log A_i=s\log i+\log f(i)+g(i)\qquad(i\geq b),$$

where $g(i)\to 0$. Therefore,

$$\sum_{i=b}^{n}\frac{\log A_i}{i}=s\sum_{i=b}^{n}\frac{\log i}{i}+\sum_{i=b}^{n}\frac{\log f(i)}{i}+\sum_{i=b}^{n}g(i)\frac{1}{i}. \tag{27}$$

Now

$$\sum_{i=b}^{n} \frac{\log i}{i} = \int_{b}^{n} \frac{\log x}{x}\, dx + O(1) = \frac{\log^2 n}{2} + O(1). \tag{28}$$

Note that the function $\frac{\log x}{x}$ is strictly decreasing from a certain value of x and the left side of (28) is a sum of rectangles of basis 1 and height $\frac{\log i}{i}$.

On the other hand, we have

$$\sum_{i=b}^{n} \frac{\log f(i)}{i} = \int_{b}^{n} \frac{\log f(x)}{x}\, dx + O(1). \tag{29}$$

Note that the integral $\int_b^\infty \frac{\log f(x)}{x}\, dx$ diverges and the function $\frac{\log f(x)}{x}$ is strictly decreasing from a certain value of x, since its derivative is negative from a certain value of x (use Equation (1)).

Note also that

$$\sum_{i=b}^{n} g(i)\frac{1}{i} = o(\log n). \tag{30}$$

Substituting (28), (29) and (30) into (27) we obtain (26). Note that (L'Hospital's rule and (1))

$$\lim_{n\to\infty} \frac{\int_b^n \frac{\log f(x)}{x}\, dx}{\log^2 n} = 0,$$

$$\lim_{n\to\infty} \frac{\int_b^n \frac{\log f(x)}{x}\, dx}{\log n} = \infty.$$

□

Theorem 1.4. *The following asymptotic formula holds*

$$\sum_{i=1}^{n} \frac{\log p_i}{i} = \frac{1}{2}\log^2 n + \log n \log\log n - \log n + o(\log n). \tag{31}$$

Proof. Here we have $f(x) = \log x$ and $s = 1$, therefore

$$\begin{aligned} \int_b^n \frac{\log f(x)}{x}\, dx &= \int_b^n \frac{\log\log x}{x}\, dx = [\log x \log\log x - \log x]_b^n \\ &= \log n \log\log n - \log n + O(1). \end{aligned} \tag{32}$$

Substituting (32) into (26) we obtain (31). □

Now, we establish the following corollary to Theorem 1.4.

Corollary 1.5. *The following limit holds*

$$\lim_{n\to\infty} \frac{\left(\prod_{i=1}^{n} p_i^{\frac{1}{i}}\right)^{\frac{1}{\log n}}}{\sqrt{n}\log n} = \frac{1}{e}.$$

Proof. We have

$$\prod_{i=1}^{n} p_i^{\frac{1}{i}} = \exp\left(\sum_{i=1}^{n} \frac{\log p_i}{i}\right).$$

□

We have the following theorem.

Theorem 1.6. *Let A_n be a strictly increasing sequence of positive integers such that*

$$A_n \sim n^s f(n) \qquad (A_1 > 1), \tag{33}$$

where $f(x)$ is a function of slow increase and s is a positive real number.

The following limit holds

$$\lim_{n\to\infty} \frac{\sqrt[n]{\log A_1 \log A_2 \ldots \log A_n}}{\log A_n} = 1. \tag{34}$$

Proof. We can suppose that a is a positive integer such that $a \geq 3$ and $f(x) > 0$ in the interval $[a, \infty]$. We have (see (33))

$$A_i \sim i^s f(i).$$

Therefore,

$$\log A_i = s \log i + \log f(i) + o(1) = s \log i \left(1 + \frac{\log f(i)}{s \log i} + \frac{o(1)}{s \log i}\right). \tag{35}$$

Now (L'Hospital's rule) we have (see (1))

$$\lim_{x\to\infty} \frac{\log f(x)}{\log x} = \lim_{x\to\infty} \frac{x f'(x)}{f(x)} = 0. \tag{36}$$

Equations (35) and (36) give

$$\log\log A_i = \log s + \log\log i + g(i), \tag{37}$$

where $g(i) \to 0$.

Now, we have (see (37))

$$\begin{aligned} \log\left(\log A_1 \log A_2 \cdots \log A_n\right) &= \sum_{i=1}^{n} \log\log A_i = \sum_{i=1}^{a-1} \log\log A_i \\ + \quad \sum_{i=a}^{n} (\log s + \log\log i + g(i)) &= n \log s + O(1) + \sum_{i=a}^{n} \log\log i \\ + \quad \sum_{i=a}^{n} g(i). \end{aligned} \tag{38}$$

Besides, we have

$$\sum_{i=a}^{n} \log\log i = \int_a^n \log\log x \, dx + O(\log\log n). \tag{39}$$

Note that the left side of (39) is a sum of rectangles of basis 1 and height $\log\log i$. On the other hand, $\log\log x$ is a function positive and strictly increasing in the interval $[a,\infty)$.

Now, we have (use integration by parts)

$$\int_a^n \log\log x\, dx = n\log\log n + O(1) - \int_a^n \frac{1}{\log t}\, dt. \tag{40}$$

On the other hand (L'Hospital's rule), we have the limit

$$\lim_{x\to\infty} \frac{\int_a^x \frac{1}{\log t}\, dt}{x} = \lim_{x\to\infty} \frac{1}{\log x} = 0. \tag{41}$$

Equations (39), (40) and (41) give

$$\sum_{i=a}^{n} \log\log i = n\log\log n + o(n). \tag{42}$$

Given $\epsilon > 0$, there exist n_0 such that if $n \geq n_0$ we have $|g(i)| < \epsilon$ (see (37)). Therefore,

$$\left|\sum_{i=a}^{n} g(i)\right| \leq \sum_{i=a}^{n} |g(i)| \leq \sum_{i=a}^{n_0-1} |g(i)| + \epsilon\sum_{i=n_0}^{n} 1 \leq \sum_{i=a}^{n_0-1} |g(i)| + \epsilon n. \tag{43}$$

Hence (see (43)) from a certain value of n we have

$$\left|\frac{\sum_{i=a}^{n} g(i)}{n}\right| \leq \frac{\sum_{i=a}^{n_0-1} |g(i)|}{n} + \epsilon \leq 2\epsilon.$$

That is

$$\sum_{i=a}^{n} g(i) = o(n). \tag{44}$$

Equations (38), (42) and (44) give

$$\log\left(\log A_1 \log A_2 \cdots \log A_n\right) = n\log\log n + n\log s + o(n). \tag{45}$$

Now, we have (see (45) and (37))

$$\begin{aligned} &\log\left(\frac{\sqrt[n]{\log A_1 \log A_2 \ldots \log A_n}}{\log A_n}\right) = \frac{1}{n}\log\left(\log A_1 \log A_2 \cdots \log A_n\right) \\ -\ &\log\log A_n = o(1). \end{aligned}$$

That is, limit (34). □

For example, If we consider the sequence $c_{n,k}$ then limit (34) becomes

$$\lim_{n\to\infty} \frac{\sqrt[n]{\log c_{1,k} \log c_{2,k} \ldots \log c_{n,k}}}{\log c_{n,k}} = 1.$$

In particular for the sequence $p_n = c_{n,1}$ of prime numbers we have

$$\lim_{n\to\infty} \frac{\sqrt[n]{\log p_1 \log p_2 \ldots \log p_n}}{\log p_n} = 1.$$

Theorem 1.7. *Let $f(x) > 0$ a function of slow increase in the interval $[a, \infty)$, where a is a positive integer. The following limit holds*

$$\lim_{n\to\infty} \frac{\sqrt[n]{f(b)f(b+1)\dots f(n)}}{f(n)} = 1, \tag{46}$$

where $b \geq a$ and b is a arbitrary but fixed positive integer.

Proof. Without loss of generality we can suppose that $f(b) \geq 1$. We have

$$\log f(b) + \log f(b+1) + \dots + \log f(n) = \int_b^n \log f(x)\, dx + O(\log f(n)). \tag{47}$$

Now (integration by parts)

$$\int_b^n \log f(x)\, dx = n \log f(n) + O(1) - \int_b^n \frac{x f'(x)}{f(x)}\, dx. \tag{48}$$

If the integral in (48) diverges we have (L'Hospital's rule and see (1))

$$\lim_{x\to\infty} \frac{\int_b^x \frac{t f'(t)}{f(t)}\, dt}{x} = \lim_{x\to\infty} \frac{x f'(x)}{f(x)} = 0. \tag{49}$$

Therefore

$$\lim_{n\to\infty} \frac{\int_b^n \frac{x f'(x)}{f(x)}\, dx}{n} = 0. \tag{50}$$

Clearly (50) also holds if the integral in (48) converges.
We also have the following limit (L'Hospital's rule and see (1))

$$\lim_{x\to\infty} \frac{\log f(x)}{\log x} = \lim_{x\to\infty} \frac{x f'(x)}{f(x)} = 0. \tag{51}$$

Equations (47), (48), (50) and (51) give

$$\log f(b) + \log f(b+1) + \dots + \log f(n) = n \log f(n) + o(n). \tag{52}$$

Equation (52) gives

$$\log \left(\frac{\sqrt[n]{f(b)f(b+1)\dots f(n)}}{f(n)} \right) = o(1). \tag{53}$$

Equation (53) gives (46). □

For example, the function $\log x$ is a function of slow increase. Therefore limit (46) becomes

$$\lim_{n\to\infty} \frac{\sqrt[n]{\log 2 \log 3 \dots \log n}}{\log n} = 1.$$

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