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Sum of dilates of two sets

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Abstract: Let $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Z}$ be nonempty finite sets and let r be a nonzero integer. The sum of dilates of A and B is defined as $A + r \cdot B := \{a + rb : a \in A \text{ and } b \in B\}$. Finding nontrivial lower bound for the sum of dilates is an important problem in additive combinatorics and it has applications in sum-product problems. In case of $A = B$, a recent result of Freiman et al. states that if $r \geq 3$, then $|A + r \cdot A| \geq 4|A| - 4$. We generalize this result for the sum of dilates $A + r \cdot B$ for two sets A and B, where r is an integer with $|r| \geq 3$.
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1 Introduction

Let A and B be nonempty finite subsets of an additive group G, and let r be a nonzero integer. As usual, we define

$$A + B := \{a + b : a \in A \text{ and } b \in B\}.$$

The *r-dilate of the set A* is defined as

$$r \cdot A := \{ra : a \in A\}.$$

For nonzero integers m and n, we can express the sum of dilates of A and B as

$$m \cdot A + n \cdot B = \{ma + nb : a \in A \text{ and } b \in B\}.$$

Finding nontrivial lower bound for the cardinality of sum of dilates is one of the important problems in additive combinatorics. Here we shall mainly consider the sum of dilates $A + r \cdot B$ of two sets A and B. The sets A and B will be usually nonempty finite subsets of the group $G = \mathbb{Z}$, if not specified. If $r = 1$, then the sumset $A + r \cdot B$ is the usual *Minkowski sum* of A and B, and there are several direct and inverse results available for this sumset in literatures. The simplest one is the following result.

Theorem A (See [14], Theorem 1.4 and Lemma 1.3)**.** *Let $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Z}$ be nonempty finite sets. Then*

$$|A + B| \geq |A| + |B| - 1.$$

The equality holds if and only if the sets A and B are arithmetic progressions with the same common difference, provided that $|A| \geq 2$ and $|B| \geq 2$.

For detailed study, the interested readers are referred to Nathanson's excellent text [14].

The study of sum of dilates has been the active area of research during last decade because of its applications in the study of sum-product problems in finite fields [8]. The study of sum of dilates dates back to 2002, when Hamidoune and Plagne [9] proved that $|A + r \cdot A| \geq 3|A| - 2$ for $|r| \geq 2$ as a byproduct of an extension of Freiman's $3k - 3$ theorem. Cilleruelo et al. [5] and Freiman et al. [7] proved some inverse results for $r = 2$. Cilleruelo et al. [5] also settled completely the direct and inverse problems for the case $r = 3$. Nathanson [15] obtained the refined lower bound $\left\lfloor \frac{7|A|-5}{2} \right\rfloor$ for $|m \cdot A + n \cdot A|$, where $m \geq 3$ and n are positive coprime integers. The sum of multiple dilates was considered by Bukh [3] and recently by Shakan [19]. Bukh's result has applications in the study of sum-product problems in finite fields [8]. Motivated by this, some refined results were obtained by Cilleruelo et al. [4], Du et al. [6], Hamidoune et al. [10] and Ljujić [13]. More recently, Balog and Shakan [1] almost completely settled the problem for the sum of dilates $m \cdot A + n \cdot A$ and proved the following result.

Theorem B. *Let m and n be coprime positive integers, where $1 \leq m < n$. Let $A \subseteq \mathbb{Z}$ be a nonempty finite set. Then*

$$|m \cdot A + n \cdot A| \geq (m + n)|A| - (mn)^{(m+n-3)(m+n)+1}.$$

All these results give lower bounds for sufficiently large sets only. Freiman et al. in their recent paper [7] obtained the uniform lower bound for $A + r \cdot A$ for all $r \geq 3$. The precise statement is the following.

Theorem C. *Let $A \subseteq \mathbb{Z}$ be a nonempty finite set of integers, and let $r \geq 3$ be an integer. Then*

$$|A + r \cdot A| \geq 4|A| - 4.$$

All the above results deal with the sum of dilates of a single set. In this paper, we prove a result similar to Theorem C for the sum of dilates of two sets A and B. We begin with some notation. Let $A = \{a_0, a_1, \ldots, a_{k-1}\} \subseteq \mathbb{Z}$ be a finite set, where $a_0 < a_1 < \cdots < a_{k-1}$. The length of A is defined by

$$\ell(A) := \max(A) - \min(A).$$

If $k \geq 2$, then we define

$$d(A) := \gcd(a_1 - a_0, a_2 - a_0, \ldots, a_{k-1} - a_0).$$

The following theorem is the main result of this paper.

Theorem 1.1. *Let r be an integer with $|r| \geq 3$. Let $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Z}$ be nonempty finite sets satisfying the following conditions:*

(i) $|A| \leq |B|$ *and* $\ell(A) \leq \ell(B)$*;*

(ii) $d(A) = d(B) = 1$ *if* $|A| \geq 2$ *and* $|B| \geq 2$*.*

Then

$$|A + r \cdot B| \geq 4|A| - 4.$$

It is not hard to see that Theorem C follows easily from Theorem 1.1. We shall prove Theorem 1.1 in Section 2. The following corollary of Theorem 1.1 extends Theorem C for negative values of r.

Corollary 1.1. *Let $A \subseteq \mathbb{Z}$ be a nonempty finite set of integers, and let r be an integer with $|r| \geq 3$. Then*

$$|A + r \cdot A| \geq 4|A| - 4.$$

With regard to the problem of estimating the lower bound for the cardinality of the sum of dilates in groups other than $\mathbb{Z}$, not much is known except for some results in cyclic groups of prime order due to Plagne [16] and Pontiveros [18]; some results on sets of small doubling in linear spaces over $\mathbb{R}$ or $\mathbb{Q}$ due to Konyagin and Laba [11]; some results on sum of dilates in $\mathbb{Z}^n$ due to Balog and Shakan [2] and in linearly orderable groups due to Plagne and Tringali [17].

2 Proof of the main theorem

The proof of Theorem 1.1 is based on Theorem A and the following theorem due to Lev, Smeliansky [12] and Stanchescu [20]. Let $\mathbb{N}_0$ denote the set of nonnegative integers.

Theorem 2.1 (See Theorem LSS in [7])**.** *Let A and B be finite subsets of $\mathbb{N}_0$ such that $0 \in A \cap B$. Define*

$$\delta_{A,B} := \begin{cases} 1, & \text{if } \ell(A) = \ell(B); \\ 0, & \text{if } \ell(A) \neq \ell(B). \end{cases}$$

Then the following statements hold:

(i) If $\ell(A) = \max(\ell(A), \ell(B)) \geq |A| + |B| - 1 - \delta_{A,B}$ *and* $d(A) = 1$*, then*

$$|A + B| \geq |A| + 2|B| - 2 - \delta_{A,B}.$$

(ii) If $\max(\ell(A), \ell(B)) \leq |A| + |B| - 2 - \delta_{A,B}$*, then*

$$|A + B| \geq \max(\ell(A) + |B|, \ell(B) + |A|).$$

Proof of Theorem 1.1. Since in case of $r < 0$, we can replace B by $B' = (-1) \cdot B$, it suffices to prove the theorem for $r > 0$ only. The result is obvious for $|A| = 1$. Now let $|A| = 2$ and let B be a nonempty set with $|B| \geq |A|$ and $\ell(A) \leq \ell(B)$. By writing $A = \{a_1, a_2\}$, $B = \{b_1, b_2, \ldots, b_n\}$, where $a_1 < a_2$ and $b_1 < b_2 < \cdots < b_n$, it is easy to see that

$$\{a_1 + rb_1 < a_2 + rb_1 < a_1 + rb_n < a_2 + rb_n\} \subseteq A + r \cdot B.$$

Therefore, $|A + r \cdot B| \geq 4 = 4|A| - 4$. Thus the theorem is true for $|A| = 2$ also.

Now assume that $|A| \geq 3$ and $|B| \geq 3$. Since the translations of A and B do not change the cardinality of $A + r \cdot B$, we may assume, without loss of generality, that $A \subseteq \mathbb{N}_0$, $B \subseteq \mathbb{N}_0$ and $0 \in A \cap B$. Let $A_1, \ldots, A_s$ be disjoint subsets of A contained in distinct residue classes modulo r such that $A = A_1 \cup \cdots \cup A_s$. Since $|A| \geq 3$, $d(A) = 1$ and $0 \in A$, it follows that $s \geq 2$. Now

$$|A + r \cdot B| = \sum_{i=1}^{s} |A_i + r \cdot B| \geq \sum_{i=1}^{s} (|A_i| + |r \cdot B| - 1) = |A| + s(|B| - 1).$$

If $s \geq 3$, then $|A + r \cdot B| \geq |A| + 3(|B| - 1) \geq 4|A| - 4$ and the result follows. Therefore, we assume that $s = 2$, so that $A = A_1 \cup A_2$, where A_1 and A_2 are nonempty and $A_1 \cap A_2 = \emptyset$. Furthermore, we can assume that $|B| < \frac{3}{2}|A| - 1$, because if $|B| \geq \frac{3}{2}|A| - 1$, then

$$|A + r \cdot B| \geq |A| + s(|B| - 1) \geq |A| + 2\left(\frac{3}{2}|A| - 1 - 1\right) = 4|A| - 4.$$

Let $|A_1| = k_1$, $|A_2| = k_2$ and $|A| = k$ so that $k_1 \geq 1$, $k_2 \geq 1$ and $k = k_1 + k_2$. Without loss of generality, we may assume that $k_1 \geq k_2$. Since $k \geq 3$, it follows that $k_1 \geq 2$. Now define

$$A_1^* = \frac{1}{r} \cdot (A_1 - \min(A_1)) \text{ and } A_2^* = \frac{1}{r} \cdot (A_2 - \min(A_2)).$$

Clearly, we have

$$|A_i^*| = |A_i| \text{ and } |A_i + r \cdot B| = |A_i^* + B| = |B + A_i^*| \text{ for } i = 1, 2.$$

Thus

$$|A + r \cdot B| = |A_1 + r \cdot B| + |A_2 + r \cdot B| = |B + A_1^*| + |B + A_2^*|.$$

Note that $\ell(A_i) \geq r(k_i - 1)$ and $\ell(A_i^*) = \frac{1}{r}\ell(A_i)$, and so

$$\ell(A_i^*) = \frac{1}{r}\ell(A_i) \leq \ell(A_i) \leq \ell(A) \leq \ell(B).$$

It can be easily verified that $\ell(A_i) > \ell(A_i^*)$ if and only if $k_i \geq 2$. It is also easy to see that $\max(\ell(B) + |A_i^*|, \ell(A_i^*) + |B|) = \ell(B) + |A_i^*|$ for $i = 1, 2$.

Case 1. Suppose that $k_1 = k - 1$ and $k_2 = 1$.

Since $k_1 \geq 2$, we have $\ell(B) \geq \ell(A_1) > \ell(A_1^*)$ and $\ell(B) \geq \ell(A_1) \geq r(k_1 - 1) \geq 3k_1 - 3$. First assume that $\ell(B) \geq |B| + |A_1^*| - 1$. Since $d(B) = 1$, it follows by Theorem 2.1(i)that $|B + A_1^*| \geq |B| + 2|A_1^*| - 2$. Also $|B + A_2^*| \geq |B|$. Therefore, it follows easily that $|A + r \cdot B| \geq 4k - 4$. In case of $\ell(B) \leq |B| + |A_1^*| - 2$, by Theorem 2.1(ii), we have $|B + A_1^*| \geq \ell(B) + |A_1^*| \geq 4k_1 - 3$. Thus again, it is easy to see that $|A + r \cdot B| \geq 4k - 4$.

Case 2. Suppose that $k_1 \geq k_2 \geq 2$.
In this case, we have $\ell(B) \geq \ell(A) \geq \ell(A_i) > \ell(A_i^*)$ and $\ell(B) \geq \ell(A) \geq \ell(A_i) \geq r(k_i - 1) \geq 3k_i - 3$ for $i = 1, 2$, Now we consider three distinct subcases.
Subcase 2(i). Suppose that $\ell(B) \geq |B| + |A_1^*| - 1$. Then $\ell(B) \geq |B| + |A_2^*| - 1$ also. Since $d(B) = 1$, it follows from Theorem 2.1(i) that

$$|B + A_i^*| \geq |B| + 2|A_i^*| - 2 \geq |A| + 2|A_i^*| - 2 = k + 2k_i - 2 \text{ for } i = 1, 2.$$

Therefore,

$$|A + r \cdot B| = |B + A_1^*| + |B + A_2^*| \geq (k + 2k_1 - 2) + (k + 2k_2 - 2) = 4k - 4.$$

Subcase 2(ii). Suppose that $|B| + |A_2^*| - 1 \leq \ell(B) \leq |B| + |A_1^*| - 2$. Then $|A_1^*| - 2 \geq |A_2^*| - 1$ which implies $k_1 \geq k_2 + 1$. By Theorem 2.1, we have

$$|B + A_1^*| \geq \ell(B) + |A_1^*| \geq \ell(A) + |A_1^*| \geq 3k_1 - 3 + k_1 = 4k_1 - 3,$$

and

$$|B + A_2^*| \geq |B| + 2|A_2^*| - 2 \geq k + 2k_2 - 2 \geq k_1 + 3k_2 - 2 \geq 4k_2 - 1.$$

Therefore,

$$|A + r \cdot B| = |B + A_1^*| + |B + A_2^*| \geq 4k_1 - 3 + 4k_2 - 1 = 4k - 4.$$

Subcase 2(iii). Suppose that $\ell(B) \leq |B| + |A_2^*| - 2$. Then we have also $\ell(B) \leq |B| + |A_1^*| - 2$. Recall that $\ell(B) \geq \ell(A) > \ell(A_i^*)$ for $i = 1, 2$. It follows by Theorem 2.1(ii) that

$$|A + r \cdot B| = |B + A_1^*| + |B + A_2^*| \geq \ell(B) + |A_1^*| + \ell(B) + |A_2^*| = 2\ell(B) + k.$$

Now if $k_1 = k_2 = \frac{k}{2}$, then

$$\ell(B) \leq |B| + |A_2^*| - 2 < \frac{3}{2}|A| - 1 + |A_2^*| - 2 = \frac{3}{2}k + k_2 - 3 = 2k - 3.$$

Therefore,

$$\frac{3}{2}k - 3 = 3k_i - 3 \leq \ell(A_i) \leq \ell(A) \leq \ell(B) \leq 2k - 4 \text{ for } i = 1, 2.$$

We claim that $\ell(B) \geq \frac{3}{2}k - 2$. Indeed, if $\ell(B) \leq \frac{3}{2}k - 3$, then $\ell(A_i) = \ell(A) = \max(A)$ for $i = 1, 2$. But $\max(A) \notin A_i$ for some i and hence $\ell(A_i) < \ell(A)$ for some i, a contradiction. This proves our claim. Therefore,

$$|A + r \cdot B| \geq 2\ell(B) + k \geq 2\left(\frac{3}{2}k - 2\right) + k = 4k - 4.$$

If $k_1 \neq k_2$, then it is easy to see that $k_1 \geq \frac{k+1}{2}$. Therefore,

$$|A + r \cdot B| \geq 2\ell(B) + k \geq 2(3k_1 - 3) + k \geq 6\left(\frac{k+1}{2}\right) + k - 6 \geq 4k - 4.$$

Thus the result is true in Case 2 also. This completes the proof. □

The following example shows that the lower bound in Theorem 1.1 can not be improved under the conditions of that theorem.

Example: Let $r \geq 3$ be a fixed integer. Let $A = \{ir, jr+1 : i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, m-2\}$ and $B = \{0, 1, \ldots, 2m-2, nr\}$, where $n \geq m \geq 3$. Then the sets A and B satisfy the conditions of Theorem 1.1. To compute $|A + r \cdot B|$, we follow the same procedure as described in the above proof. In this case, we have $A_1^* = \{0, 1, \ldots, m\}$ and $A_2^* = \{0, 1, \ldots, m-2\}$. It can be easily verified that

$$B + A_1^* = \{0, 1, \ldots, 3m-2, nr, nr+1, \ldots, nr+m\},$$

and

$$B + A_2^* = \{0, 1, \ldots, 3m-4, nr, nr+1, \ldots, nr+m-2\}.$$

Therefore,

$$|A + r \cdot B| = |B + A_1^*| + |B + A_2^*| = 4m + 4m - 4 = 8m - 4 = 4|A| - 4.$$

Thus the lower bound is best possible in Theorem 1.1.

3 Concluding remarks

If $|A| \geq 2$ and $d(A) = 1$, then $s \geq 2$, as shown in the proof of Theorem 1.1. Therefore, $|A + r \cdot B| \geq |A| + s(|B| - 1) \geq |A| + 2(|B| - 1) = |A| + 2|B| - 2$. Now if $|B| \geq \frac{3}{2}|A| - 1$, then clearly we have $|A + r \cdot B| \geq 4|A| - 4$. Thus Theorem 1.1 remains true in this case without the assumptions $\ell(A) \leq \ell(B)$ and $d(B) = 1$. In fact, in this case the lower bound $|A| + 2|B| - 2$ is better than $4|A| - 4$ and this lower bound is also in terms of both $|A|$ and $|B|$.

The following remarks show that the conditions (i) and (ii) of Theorem 1.1 are sufficient but not necessary.

Remark 3.1. *The conclusion of Theorem 1.1 may or may not hold even if $|A| \leq |B|$ and $d(A) = d(B) = 1$, but $\ell(A) > \ell(B)$, which can be seen by considering the following examples, respectively:*

(i) Let $r \geq 3$ be fixed. Let $A = \{0, r, 2r, \ldots, (m-1)r\} \cup \{1, r+1, 2r+1, \ldots, mr+1\}$ and $B = \{0, 1, \ldots, 3m-2\} \cup \{3m\}$, where $m \geq 3$. Then by splitting A into distinct residue classes modulo r and taking sum of each part with B, one can easily verify, using Theorem A, that $|A + r \cdot B| = 8m + 1 > 8m = 4|A| - 4$.

(ii) Let $r \geq 3$ be fixed. Let $A = \{0, r, 2r, \ldots, mr\} \cup \{1, r+1, 2r+1, \ldots, (m-1)r+1\}$ and $B = \{0, 1, \ldots, 3m-1\}$, where $m \geq 2$. Then one can easily verify that $|A + r \cdot B| = 8m - 1 < 8m = 4|A| - 4$.

Remark 3.2. *The conclusion of Theorem 1.1 may or may not hold even if $|A| \leq |B|$ and $\ell(A) \leq \ell(B)$, but $d(A) \neq 1$, which can be seen by considering the following examples, respectively:*

(i) Let $r \geq 3$ be fixed and let $k \geq 2$ be an integer such that $\gcd(k, r) = 1$. Let $A = k \cdot \{0, r, 2r, \ldots, (m-1)r\} \cup k \cdot \{1, r+1, 2r+1, \ldots, mr+1\}$ and $B = k \cdot \{0, 1, \ldots, 3m-3\} \cup \{1, k(2mr+1)\}$, where $m \geq 3$. Then by splitting A into distinct residue classes modulo r and taking sum of each part with $B' = B \setminus \{1\}$, one can easily verify that that $|A + r \cdot B| \geq |A + r \cdot B'| = 10m - 4 > 8m = 4|A| - 4$.

(ii) Let $r \geq 3$ be fixed. Let $A = \{0, r, 2r, \ldots, (k-1)r\}$ and $B = \{0, 1, \ldots, k-1\} \cup \{rk\}$, where $k \geq 5$. Then one can easily verify that $|A + r \cdot B| = 3k - 1 < 4k - 4 = 4|A| - 4$.

Remark 3.3. *The conclusion of Theorem 1.1 may or may not hold even if $d(A) = d(B) = 1$ and $\ell(A) \leq \ell(B)$, but $|A| > |B|$ which can be seen by considering the following examples, respectively:*

(i) Let $r \geq 3$ be fixed. Let $A = \{0, r, 2r, \ldots, mr\} \cup \{1, r+1, 2r+1, \ldots, (m-1)r+1\}$ and $B = \{0, 1, \ldots, 2m-3\} \cup \{mr, mr+m\}$, where $m \geq 2$. Then one can easily verify that $|A + r \cdot B| \geq 10m - 4 \geq 8m = 4|A| - 4$.

(ii) Let $r \geq 3$ be fixed. Let $A = \{0, r, 2r, \ldots, mr\} \cup \{1, r+1, 2r+1, \ldots, (m-1)r+1\}$ and $B = \{0, 1, \ldots, m\} \cup \{mr\}$, where $m \geq 3$. Then one can easily verify that $|A + r \cdot B| = 6m + 2 < 8m = 4|A| - 4$.

These observations raise the following questions:

1. Can we improve the lower bound in Theorem 1.1 under some other conditions on the sets A and B ?
2. More generally, can we obtain the result similar to Theorem B for the sum of dilates $m \cdot A + n \cdot B$, where m and n are coprime integers?

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