Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 23, 2017, No. 4, 18–21

A note on bounds for the Neuman–Sándor mean using power and identric means

József Sándor

Department of Mathematics, Babeş–Bolyai University
Cluj-Napoca, Romania

Received: 8 February **Accepted:** 29 October 2017

Abstract: In this note we point out priority results and new proofs related to the bounds for the Neuman–Sándor mean in terms of the power means and the identric means.
Keywords: Bounds, Neuman–Sándor mean, Identric mean, Power mean.
AMS Classification: 26E60.

1 Introduction

For $k \in \mathbb{R}$ the k-th power mean $A_p(a,b)$, Neuman–Sándor Mean $M(a,b)$ [4] and the identric mean $I(a,b)$ of two positive real numbers a and b are defined by

$$A_k(a,b) = \left(\frac{a^k + b^k}{2}\right)^{1/k} (k \neq 0); A_0(a,b) = \sqrt{ab} = G(a,b) \tag{1}$$

$$M(a,b) = \frac{a-b}{2\text{arcsinh}\big((a-b)(a+b)\big)} (a \neq b); M(a,a) = a \tag{2}$$

$$I(a,b) = \frac{1}{e}\big(b^b/a^a\big)^{1/(b-a)} (a \neq b); I(a,a) = a \tag{3}$$

respectively, where $\text{arcsinh}(x) = \log(x + \sqrt{1+x^2})$ denotes the inverse hyperbolic sine function.

While the kth power means and the identric mean have been studied extensively in the last 30–40 years (see e.g. [7] or [2] for surveys of results), the Neuman–Sándor mean has been introduced in 2003 [4] and studied also in 2006 [5], as a particular Schwab–Borchardt mean. In the last 10 years, the Neuman–Sándor mean has been studied by many authors, for many references, see e.g. the papers [10] and [11], [3].

In 2012 and 2013, independently Z.-H. Yang [10] and Y.-M. Chu, B.-Y. Long [3] have considered the bounds

$$A_r < M < A_{4/3}, \tag{4}$$

where $M = M(a, b)$ for $a \neq b$, etc; and $r = \dfrac{\log 2}{\log\log(3 + 2\sqrt{2})} = 1.244\ldots$ Also, the constants r and $4/3$ are best possible.Though not mentioned explicitely, the upper bound of (4) is due to E. Neuman and J. Sándor. Indeed, they proved the strong inequalities (see also [3]):

$$M(a, b) < \frac{2A + Q}{3} < \left[He(a^2, b^2)\right]^{1/2} < A_{4/3}(a, b), \tag{5}$$

where $He(x, y) = \dfrac{x + \sqrt{xy} + y}{3}$ denotes the Heronian mean and

$$A = A(a, b) = A_1(a, b); Q = Q(a, b) = \left(\frac{a^2 + b^2}{2}\right)^{1/2} = A_2(a, b).$$

The first inequality of (5) appears in [4], while the second one results by remarking that $He(a^2, b^2) = \dfrac{2a^2 + b^2}{3} = \dfrac{Q^2 + 2A^2}{3}$ and the fact that $\dfrac{Q^2 + 2A^2}{3} > \left(\dfrac{2A + Q}{3}\right)^2$.

The last inequality of (5) follows from

$$He(a, b) < A_{2/3}(a, b) \tag{6}$$

(see [9], [6]) applied to $a := a^2, b := b^2$.

We note also that for application purposes, we may choose $1.2 = \dfrac{6}{5}$ in place of r in (4), so the following bounds (though, the lower bound slightly weaker) may be stated:

$$A_{6/5} < M < A_{4/3} \tag{7}$$

In the recent paper [3], M is compared also to the identric mean I, in the following manner:

$$1 < \frac{M}{I} < c, \tag{8}$$

where $c = \dfrac{e}{2\log(1 + \sqrt{2})}$ and $M = M(a, b)$ for $a \neq b$; etc.

Also, the constants 1 and c in (8) follows from earlier known results. Also, the optimality of constants follows from the proofs of these known results.

2 Main results

In [4] it is shown that

$$1 < \frac{M}{A} < \frac{1}{\text{arcsinh}(1)} = \frac{1}{\log(1 + \sqrt{2})}, \tag{9}$$

where $M = M(a,b)$ for $a \neq b$; etc.

Now, by a result of H. Alzer [1] one has

$$1 < \frac{A}{I} < \frac{e}{2} \tag{10}$$

We note that inequality (10) has been rediscovered many times. See e.g. the author's papers [6], [8].

Now, by a simple multiplication of (9) and (10), we get (8).

For the proof of the fact that 1 and c are best possible, we shall use the proofs of (9) and (10) from [4] resp. [6]. In [4] it is shown that

$$\frac{M}{A} = \frac{z}{\text{arcsinh} z}, \text{ where } z = \frac{b-a}{b+a}. \tag{11}$$

Let $b > a$. Then the function

$$f_1(z) = \frac{z}{\text{arcsinh} z}$$

is strictly increasing in $(0,1)$. Put $\frac{b}{a} = x$. Then $z = \frac{x-1}{x+1}$ is a strictly increasing function of $x > 1$. Therefore $f_1(z)$, as a composite function, will be strictly increasing also on $x \in (1, +\infty)$.

For the proof of (10) in [6] it is shown that

$$f_2(x) = \frac{A(x,1)}{I(x,1)}$$

is strictly increasing of $x > 1$.

Now, remarking that

$$\frac{M(x,1)}{I(x,1)} = f_1\big(z(x)\big) \cdot f_2(x) = g(x),$$

from the above, we get that $g(x)$ is a strictly increasing function, as the product of two functions having the same property. This gives

$$\lim_{x \to 1} g(x) < g(x) < \lim_{x \to \infty} g(x).$$

As $\lim_{x \to 1} g(x) = \lim_{x \to 1} f_1(z(x)) \cdot \lim_{x \to 1} f_2(x) = 1$ and $\lim_{x \to \infty} g(x) = \lim_{x \to \infty} f_1(z(x)) \cdot \lim_{x \to \infty} f_2(x) = \frac{1}{\log(1+\sqrt{2})} \cdot \frac{e}{2} = c$ we get the optimality of the constants from (8).

We note that the proof of (8) given in [3] is complicated, and based on subsequent derivatives of functions.

References

[1] Alzer, H. (1988), Aufgabe 987, *Elemente der Mathematik*, 43, 93.

[2] Bullen, P. S. (2003), *Handbook of means and their inequalities*, Kluwer Acad. Publ.

[3] Chu, Y.-M. & Long, B.-Y. (2013), Bounds of the Neuman–Sándor mean using power and identric means, *Abstr. Appl. Anal.*, 2013, ID 832591, 6 pages.

[4] Neuman, E. & Sándor, J. (2003) On the Schwab–Bachardt mean, *Mathematica Pannonica*, 14(2), 253–266.

[5] Neuman, E. & Sándor, J. (2006) On the Schwab–Bachardt mean II, *Mathematica Pannonica*, 17(1), 49–59.

[6] Neuman, E. & Sándor, J. (2009) Companion inequalities for certain bivariate means, *Applicable Analysis Discr. Math.*, 3, 46–51.

[7] Sándor, J. (1990) On the identric and logarithmic means, *Aequationes Mathematicae*, 40(1), 261–270.

[8] Sándor, J. (1995) On certain inequalities for means, *J. Math. Anal. Appl.*, 189, 602–606.

[9] Sándor, J. (2001) On certain inequalities for means, III, *Archiv der Mathematik*, 76(1), 34–40.

[10] Yang, Z.-H. (2012) Sharp power mean bounds for Neuman–Sándor mean, `http://arxiv.org/abs/1208.0895`.

[11] Yang, Z.-H. (2013) Estimates for Neuman–Sándor mean by power means and their relative errors, *J. Math. Ineq.*, 7(4), 711–726.