Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 23, 2017, No. 4, 14–17

Short remark on a special numerical sequence

Krassimir T. Atanassov

Department of Bioinformatics and Mathematical Modelling
IBPhBME, Bulgarian Academy of Sciences
Acad. G. Bonchev Str. Bl. 105, Sofia-1113, Bulgaria
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8010, Bulgaria
e-mail: `krat@bas.bg`

Dedicated to the 200th anniversary of Ghent University

Received: 15 October 2017 **Accepted:** 29 October 2017

Abstract: The sequence $G = \{2^2 3^3 ... p_n^{p_n}\}_{n\geq 1}$ is discussed and some of its properties are studied.
Keywords: Arithmetic function, Prime number, Sequence.
AMS Classification: 11A25

1 Introduction

Let $p_1 = 2, p_2 = 3, p_3 = 5, ...$ be the sequence of the prime numbers. We define the new sequence $G = \{G_n\}_{n\geq 1}$, where

$$G_n = 2^2 3^3 \dots p_n^{p_n}$$

and study some of its properties. Here the denotation G comes from Ghent.

First, we introduce definitions of some well-known arithmetic functions, which are defined for the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $k, \alpha_1, ..., \alpha_k, k \geq 1$ are natural numbers and $p_1, ..., p_k$ are different primes, by:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1),\ \varphi(1) = 1,$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}, \ \sigma(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 1), \ \psi(1) = 1,$$

$$\omega(n) = k$$

(see, e.g. [5, 6]).

In addition, following [1], we introduce four other functions for the same values of n:

$$\delta(n) = \sum_{i=1}^{k} \alpha_i p_1^{\alpha_1} ... p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} ... p_k^{\alpha_k},$$

$$\eta(n) = \sum_{i=1}^{k} \alpha_i p_i,$$

$$\text{mult}(n) = \prod_{k=1}^{n} p_k,$$

$$\text{sum}_2(n) = \sum_{i=1}^{k} p_i^2.$$

In [2–4] are introduced, respectively, the following three functions, called irrational, converse and restrictive factors:

$$IF(n) = \prod_{i=1}^{k} p_i^{1/\alpha_i},$$

$$CF(n) = \prod_{i=1}^{k} \alpha_i^{p_i},$$

$$RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}$$

(see, also [7]).

2 Main results

First, we see directly that for each natural number n:

$$G_n > \prod_{k=1}^{n} (p_k!)$$

and

$$G_n = 2^{2\log_2 2}.2^{3\log_2 3}.2^{5\log_2 5}. \ \ldots \ .2^{p_n \log_2 p_n} = 2^{\sum\limits_{k=1}^{n} p_k \log_2 p_k}.$$

Second, for the same n:

$$\varphi(G_n) = 2.3^2.2.5^4.4.\dots p_n^{p_n-1}.(p_n-1) = \frac{1}{2}.\frac{2}{3}.\frac{4}{5}.\ \dots\ .\frac{p_n-1}{p_n}.G_n,$$

$$\psi(G_n) = 3.2.3^2.4.5^4.6.\dots p_n^{p_n-1}.(p_n+1) = \frac{3}{2}.\frac{4}{3}.\frac{6}{5}.\ \dots\ .\frac{p_n+1}{p_n}.G_n.$$

Therefore,

$$\varphi(G_n)\psi(G_n) = \frac{2^2-1}{2^2}.\frac{3^2-1}{3^2}.\frac{5^2-1}{5^2}.\ \dots\ .\frac{p_n^2-1}{p_n^2}.G_n^2.$$

Third,

$$\sigma(G_n) = \frac{2^3-1}{2-1}.\frac{3^4-1}{3-1}.\frac{5^6-1}{5-1}.\ \dots\ .\frac{p_n^{p_n+1}-1}{p_n-1}$$
$$< 2^3.3^4.5^6.\ \dots\ .p_n^{p_n+1} = G_n\text{mult}(G_n).$$

Fourth,

$$\eta(G_n) = 2.2+3.3+5.5+\dots p_n.p_n = \text{sum}_2(2.3.5.\ \dots\ .p_n) = \text{sum}_2(G_n).$$

Fifth, obviously, for each natural number $n \geq 1$

$$\omega(G_n) = n.$$

Therefore, it is valid the following assertion.

Theorem 1. *For each natural number $n \geq 1$:*

$$\delta(G_n) = \omega(G_n).G_n.$$

Proof. Let $n \geq 1$ be a natural number. Then

$$\delta(G_n) = 2.2^{2-1}.3^3.\ \dots\ .p_n^{p_n} + 3.2^2.3^{3-1}.\ \dots\ .p_n^{p_n} + p_n.2^2.3^3.\ \dots\ .p_n^{p_n-1}$$

$$= n.2^2.3^3.\ \dots\ .p_n^{p_n} = n.G_n = \omega(G_n).G_n.$$

Sixth, we see directly that

$$CF(G_n) = G_n.$$

Seventh,

$$RF(G_n) = 2^{2-1}.3^{3-1}.5^{5-1}.\ \dots\ .p_n^{p_n-1} = \frac{G_n}{\text{mult}(G_n)}.$$

Therefore,

$$\sigma(G_n).RF(G_n) < G_n^2.$$

Moreover,

$$\varphi(G_n)\sigma(G_n) = (2^3-1).(3^4-1).(5^6-1).\ \dots\ .(p_n^{p_n+1}-1).2.3^2.5^4.\ \dots\ .p_n^{p_n-1}$$

$$= (2^3-1).(3^4-1).(5^6-1).\ \dots\ .(p_n^{p_n+1}-1).RF(G_n).$$

□

Finally, eighth, we prove the following assertion.

Theorem 2. *For each natural number n:*

$$IF(G_n) < 2^{\omega(G_n)}.$$

Proof. From $2^s > s$ for each natural number s it follows that $\sqrt[s]{s} < 2$. Hence,

$$IF(G_n) = \sqrt[2]{2}.\sqrt[3]{3}.\sqrt[5]{5}. \ \dots \ . \sqrt[p_n]{p_n} < 2^n = 2^{\omega(G_n)}.$$

□

3 Conclusion

This short remark was written at the time of my visit in Ghent (Gent, Gand) University in October 2017, when the institution celebrated the 200th anniversary of its foundation.

All the results can obtain more precise forms (e.g., at least a part of the inequalities can obtain stronger forms) and this will be an object of a future research.

References

[1] Atanassov, K. (1987) New integer functions, related to "φ" and "σ" functions, *Bulletin of Number Theory and Related Topics*, XI(1), 3–26.

[2] Atanassov, K. (1996) Irrational factor: definition, properties and problems, *Notes on Number Theory and Discrete Mathematics*, 2(3), 42–44.

[3] Atanassov, K. (2002) Converse factor: definition, properties and problems, Notes on Number Theory and Discrete Mathematics, 8(1), 37–38.

[4] Atanassov, K. (2002) Restrictive factor: definition, properties and problems, *Notes on Number Theory and Discrete Mathematics*, 8(4), 117–119.

[5] Mitrinovic, D., & Sándor, J. (1996) *Handbook of Number Theory*, Kluwer Academic Publishers.

[6] Nagell, T. (1950) *Introduction to Number Theory*, John Wiley & Sons, New York.

[7] Panaitopol, L. (2004) Properties of the Atanassov functions. *Advanced Studies on Contemporary Mathematics*, 8(1), 55–59.