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On two new two-dimensional extensions of the Fibonacci sequence

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Abstract: Two new two-dimensional extensions of the Fibonacci sequence are introduced. Explicit formulas for their n-th members are given.
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To my colleague, coauthor and friend Janusz Kacprzyk
for his $F_9 + F_7 + F_3$ anniversary

1 Introduction

Let $\{F_i\}_{i\geq 0} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 55, ...\}$ be the Fibonacci sequence.

In [1], L. Atanassova, D. Sasselov and the author, in [2] the author and in [4] J.-Z. Lee and J.-S. Lee published (second and third papers were written independently) the first extensions of the Fibonacci sequence in the form of two sequences, i.e., they introduced the first two-dimensional extensions of these sequences (cf. [3]). Later, a lot of subsequent extensions were introduced:

three and n-Fibonacci sequences, three-dimensional sequences, etc. The first of these extensions has one of the following four forms:

$$\begin{aligned} &\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d \\ &\alpha_{n+2} = \beta_{n+1} + \beta_n,\ n \geq 0 \\ &\beta_{n+2} = \alpha_{n+1} + \alpha_n,\ n \geq 0 \end{aligned} \tag{1}$$

$$\begin{aligned} &\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d \\ &\alpha_{n+2} = \alpha_{n+1} + \beta_n,\ n \geq 0 \\ &\beta_{n+2} = \beta_{n+1} + \alpha_n,\ n \geq 0 \end{aligned} \tag{2}$$

$$\begin{aligned} &\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d \\ &\alpha_{n+2} = \beta_{n+1} + \alpha_n,\ n \geq 0 \\ &\beta_{n+2} = \alpha_{n+1} + \beta_n,\ n \geq 0 \end{aligned} \tag{3}$$

$$\begin{aligned} &\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d \\ &\alpha_{n+2} = \alpha_{n+1} + \alpha_n,\ n \geq 0 \\ &\beta_{n+2} = \beta_{n+1} + \beta_n,\ n \geq 0 \end{aligned} \tag{4}$$

Graphically, the $(n+2)$-nd members of the different schemes are obtained from the n-th and the $(n+1)$-st members as shown in Fig. 1–4.

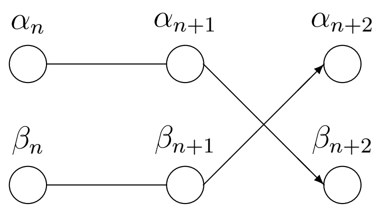

Fig. 1.

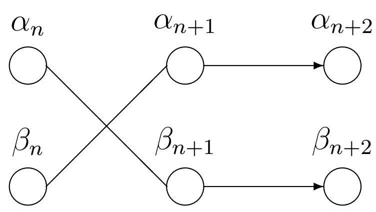

Fig. 2.

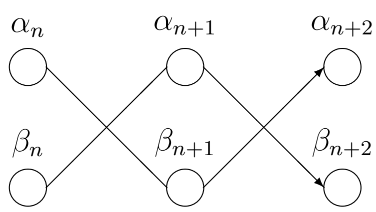

Fig. 3.

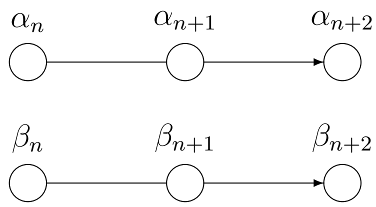

Fig. 4.

Obviously, the third and fourth two-Fibonacci sequences are two standard Fibonacci sequences, while the first two two-Fibonacci sequences are essential extensions of the standard Fibonacci sequence.

In the present paper, for the first time we construct two new types of Fibonacci sequences. For them, we show that, from one side, obviously, are more complex than the standard Fibonacci sequence, and from another, are simpler than the above discussed two-Fibonacci sequences.

2 First new scheme

Let everywhere below, a, b, c be three fixed constants. We will construct two schemes for 2-dimensional Fibonacci sequences, for which we can see that they are the simplest ones.

Let

$$\alpha_0 = \beta_0 = a, \ \ \alpha_1 = b, \ \ \beta_1 = c$$

and for each natural number $n \geq 1$:

$$\alpha_{2n} = \beta_{2n} = \alpha_{2n-2} + \frac{1}{2}(\alpha_{2n-1} + \beta_{2n-1}), \tag{1}$$

$$\alpha_{2n+1} = \alpha_{2n} + \beta_{2n-1}, \tag{2}$$

$$\beta_{2n+1} = \beta_{2n} + \alpha_{2n-1}. \tag{3}$$

The first ten members of the sequences $\{\alpha_i\}_{i\geq 0}$ and $\{\beta_i\}_{i\geq 0}$ are shown in Table 1.

Table 1

n	α_n	$\alpha_n = \beta_n$	β_n
0		a	
1	b		c
2		$a + \frac{1}{2}b + \frac{1}{2}c$	
3	$a + \frac{1}{2}b + \frac{3}{2}c$		$a + \frac{3}{2}b + \frac{1}{2}c$
4		$2a + \frac{3}{2}b + \frac{3}{2}c$	
5	$3a + 3b + 2c$		$3a + 2b + 3c$
6		$5a + 4b + 4c$	
7	$8a + 6b + 7c$		$8a + 7b + 6c$
8		$13a + \frac{21}{2}b + \frac{21}{2}c$	
9	$21a + \frac{35}{2}b + \frac{33}{2}c$		$21a + \frac{33}{2}b + \frac{35}{2}c$

The following assertion is valid

Theorem 1. For each natural number $n \geq 1$:

$$\alpha_{2n} = \beta_{2n} = F_{2n-1}a + \frac{F_{2n}}{2}b + \frac{F_{2n}}{2}c, \tag{4}$$

$$\alpha_{2n+1} = F_{2n}a + \frac{F_{2n+1} + (-1)^n}{2}b + \frac{F_{2n+1} - (-1)^n}{2}c, \tag{5}$$

$$\beta_{2n+1} = F_{2n}a + \frac{F_{2n+1} - (-1)^n}{2}b + \frac{F_{2n+1} + (-1)^n}{2}c. \tag{6}$$

Proof. Let us assume that (4)–(6) are valid for the first $2n+1$ natural numbers. Then from (1)–(6):

$$\alpha_{2n+2} = \beta_{2n+2} = \alpha_{2n} + \frac{1}{2}(\alpha_{2n+1} + \beta_{2n+1})$$

$$= \left(F_{2n-1}a + \frac{F_{2n}}{2}b + \frac{F_{2n}}{2}c\right) + \frac{1}{2}\left(F_{2n}a + \frac{F_{2n+1} + (-1)^n}{2}b + \frac{F_{2n+1} - (-1)^n}{2}c\right.$$

$$\left. + F_{2n}a + \frac{F_{2n+1} - (-1)^n}{2}b + \frac{F_{2n+1} + (-1)^n}{2}c\right)$$

$$= F_{2n+1}a + \frac{F_{2n+2}}{2}b + \frac{F_{2n+2}}{2}c;$$

$$\alpha_{2n+3} = \alpha_{2n+2} + \beta_{2n+1}$$

$$= F_{2n+1}a + \frac{F_{2n+2}}{2}b + \frac{F_{2n+2}}{2}c + F_{2n}a + \frac{F_{2n+1} - (-1)^n}{2}b + \frac{F_{2n+1} + (-1)^n}{2}c$$

$$= F_{2n+2}a + \frac{F_{2n+3} - (-1)^n}{2}b + \frac{F_{2n+3} + (-1)^n}{2}c$$

$$= F_{2n+2}a + \frac{F_{2n+3} + (-1)^{n+1}}{2}b + \frac{F_{2n+3} - (-1)^{n+1}}{2}c;$$

$$\beta_{2n+3} = \beta_{2n+2} + \alpha_{2n+1}$$

$$= F_{2n+1}a + \frac{F_{2n+2}}{2}b + \frac{F_{2n+2}}{2}c + F_{2n}a + \frac{F_{2n+1} + (-1)^n}{2}b + \frac{F_{2n+1} - (-1)^n}{2}c$$

$$= F_{2n+2}a + \frac{F_{2n+3} + (-1)^n}{2}b + \frac{F_{2n+3} - (-1)^n}{2}c$$

$$= F_{2n+2}a + \frac{F_{2n+3} - (-1)^{n+1}}{2}b + \frac{F_{2n+3} + (-1)^{n+1}}{2}c. \qquad \square$$

3 Second new scheme

Let

$$\alpha_0 = am \ \ \beta_0 = b, \ \ \alpha_1 = \beta_1 = c$$

and for each natural number $n \geq 1$:

$$\alpha_{2n} = \alpha_{2n-1} + \beta_{2n-2}, \tag{7}$$

$$\beta_{2n} = \beta_{2n-1} + \alpha_{2n-2}. \tag{8}$$

$$\alpha_{2n+1} = \beta_{2n+1} = \alpha_{2n-1} + \frac{1}{2}(\alpha_{2n} + \beta_{2n}), \tag{9}$$

The first ten members of the sequences $\{\alpha_i\}_{i\geq 0}$ and $\{\beta_i\}_{i\geq 0}$ are shown in Table 2.

Table 2

n	α_n	$\alpha_n = \beta_n$	β_n
0	a		b
1		c	
2	$b+c$		$a+c$
3		$\frac{1}{2}a + \frac{1}{2}b + 2c$	
4	$\frac{3}{2}a + \frac{1}{2}b + 3c$		$\frac{1}{2}a + \frac{3}{2}b + 3c$
5		$\frac{3}{2}a + \frac{3}{2}b + 5c$	
6	$2a+3b+8c$		$3a+2b+8c$
7		$4a+4b+13$	
8	$7a+6b+21c$		$6a+7b+21c$
9		$\frac{21}{2}a + \frac{21}{2}b + 34c$	

The following assertion is valid

Theorem 2. For each natural number $n \geq 2$:

$$\alpha_{2n-1} = \beta_{2n-1} = \frac{F_{2n-2}}{2}a + \frac{F_{2n-2}}{2}b + F_{2n-1}c, \tag{10}$$

$$\alpha_{2n} = \frac{F_{2n-1} - (-1)^n}{2}a + \frac{F_{2n-1} + (-1)^n}{2}b + F_{2n}c, \tag{11}$$

$$\beta_{2n} = \frac{F_{2n-1} + (-1)^n}{2}a + \frac{F_{2n-1} - (-1)^n}{2}b + F_{2n}c, \tag{12}$$

Proof. Let us assume that (10)–(12) are valid for the first $2n$ natural numbers. Then from (7)–(12):

$$\alpha_{2n+1} = \beta_{2n+1} = \alpha_{2n-1} + \frac{1}{2}(\alpha_{2n} + \beta_{2n})$$

$$= \frac{F_{2n-2}}{2}a + \frac{F_{2n-2}}{2}b + F_{2n-1}c + \frac{1}{2}\left(\frac{F_{2n-1} - (-1)^n}{2}a + \frac{F_{2n-1} + (-1)^n}{2}b + F_{2n}c\right.$$

$$\left. + \frac{F_{2n-1} + (-1)^n}{2}a + \frac{F_{2n-1} - (-1)^n}{2}b + F_{2n}c\right)$$

$$= \frac{F_{2n}}{2}a + \frac{F_{2n}}{2}b + F_{2n+1}c;$$

$$\alpha_{2n+2} = \alpha_{2n+1} + \beta_{2n}$$

$$= \frac{F_{2n}}{2}a + \frac{F_{2n}}{2}b + F_{2n+1}c + \frac{F_{2n-1} + (-1)^n}{2}a + \frac{F_{2n-1} - (-1)^n}{2}b + F_{2n}c$$

$$= \frac{F_{2n+1} + (-1)^n}{2}a + \frac{F_{2n+1} - (-1)^n}{2}b + F_{2n+2}c$$

$$= \frac{F_{2n+1} - (-1)^{n+1}}{2}a + \frac{F_{2n+1} + (-1)^{n+1}}{2}b + F_{2n+2}c;$$

$$\beta_{2n+2} = \beta_{2n+1} + \alpha_{2n}$$

$$= \frac{F_{2n}}{2}a + \frac{F_{2n}}{2}b + F_{2n+1}c + \frac{F_{2n-1} - (-1)^n}{2}a + \frac{F_{2n-1} + (-1)^n}{2}b + F_{2n}c$$

$$= \frac{F_{2n+1} - (-1)^n}{2}a + \frac{F_{2n+1} + (-1)^n}{2}b + F_{2n+2}c$$

$$= \frac{F_{2n+1} + (-1)^{n+1}}{2}a + \frac{F_{2n+1} - (-1)^{n+1}}{2}b + F_{2n+2}c.$$

□

4 Discussion

The two new schemes are extensions of the standard Fibonacci sequence, because for the first scheme when $b = c$, from (1)–(6) we obtain that the first ten members of the sequences $\{\alpha_i\}_{i\geq 0}$ and $\{\beta_i\}_{i\geq 0}$ are these, shown in Table 3, that, obviously, are the sequential members of the standard Fibonacci sequence.

Table 3

n	α_n	$\alpha_n = \beta_n$	β_n
0		a	
1	b		b
2		$a + b$	
3	$a + 2b$		$a + 2b$
4		$2a + 3b$	
5	$3a + 5b$		$3a + 5b$
6		$5a + 8b$	
7	$8a + 13b$		$8a + 13b$
8		$13a + 21b$	
9	$21a + 34b$		$21a + 34b$

Analogously, if $a = b$, from (7)–(12) for the second scheme we obtain that the first ten members of the sequences $\{\alpha_i\}_{i\geq 0}$ and $\{\beta_i\}_{i\geq 0}$ that are shown in Table 4.

Table 4

n	α_n	$\alpha_n = \beta_n$	β_n
0	a		a
1		c	
2	$a + c$		$a + c$
3		$a + 2c$	
4	$2a + 3c$		$2a + 3c$
5		$3a + 5c$	
6	$5a + 8c$		$5a + 8c$
7		$8a + 13$	
8	$13a + 21c$		$13a + 21c$
9		$21a + 34c$	

The geometrical interpretations of both new schemes are given in Figs. 5 and 6.

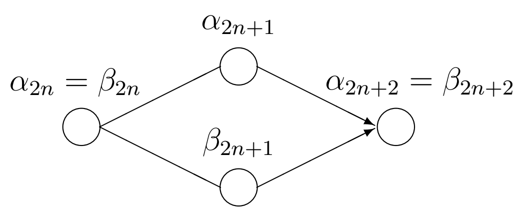

Fig. 5.

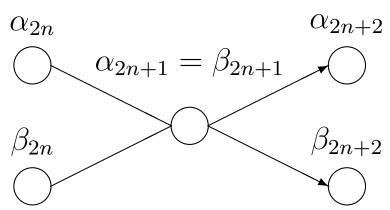

Fig. 6.

We finish with the following open problem.

Open problem: Are both schemes the simplest extensions of the standard Fibonacci sequence?

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