A short proof of a concrete sum

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Abstract: We give an alternative proof of a formula that generalizes Hermite’s identity. Instead involving modular arithmetic, our short proof relies on the Fourier-type expansion for the floor function and on a trigonometric formula.

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A closed form for \( \sum_{k=0}^{m-1} \left\lfloor \frac{x+nk}{m} \right\rfloor \), where \( x \) is a real number and \( m, n \) are integers with \( m > 0 \) can be found in [1], where the authors use modular arithmetic to establish the formula in an elementary (although somewhat long) manner. Our aim is to give a short proof of the above-mentioned result. To this end, define the fractional part of any real \( x \) by \( \{x\} = x - \lfloor x \rfloor \) and notice that it is a periodic piecewise linear function, discontinuous at each integer point, whose Fourier expansion gives us

\[
\lfloor x \rfloor = x - \frac{1}{2} + \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{\sin(2\pi j x)}{j}, \quad x \in \mathbb{R} \setminus \mathbb{Z}.
\] (1)

If \( f(x) \) stands for the right-hand side of (1), then \( f(n) = n - 1/2 \) for each integer \( n \), and thus \( \lfloor n \rfloor = n = f(n) + 1/2 \). From (1) and using that \( \sum_{k=0}^{m-1} k = (m-1)m/2 \), one gets

\[
\sum_{k=0}^{m-1} \left\lfloor \frac{x+nk}{m} \right\rfloor = x + \frac{(m-1)n}{2} - \frac{m}{2} + \frac{1}{\pi} \sum_{j=1}^{\infty} \sum_{k=0}^{m-1} \sin\left(\frac{2\pi j}{m} \frac{x+nk}{m}\right),
\] (2)

provided none of the \( (x+nk)/m \) is an integer.

By using \( \sum_{k=0}^{p} \sin(za + ak) = \csc(a/2) \sin(a(p+1)/2) \sin(z + ap/2) \) (see [2]) we establish that

\[
\sum_{k=0}^{m-1} \sin\left(2\pi j \frac{x+nk}{m}\right) = \frac{\sin(\pi j n)}{\sin(\pi j \frac{n}{m})} \sin(\pi j n - \pi j \frac{n}{m} + 2\pi j \frac{x}{m}).
\] (3)
Note that the above sum vanishes except, eventually, when the denominator at the right-hand side also vanishes. Therefore, denoting \( d = \gcd(m, n) \), \( m' = m/d \) and \( n' = n/d \), (3) may differ from zero only when \( jn/m = jn'/m' \in \mathbb{Z} \), namely, when \( j = lm' = lm/d \) (\( l = 1, 2, 3, \ldots \)).

With this in mind, (2) finally transforms to

\[
\sum_{k=0}^{m-1} \left[ \frac{x + nk}{m} \right] = x + \frac{(m-1)n}{2} - \frac{m}{2} + \frac{d}{\pi} \sum_{l=1}^{\infty} \sum_{k=0}^{m-1} \frac{\sin \left( 2\pi ln'k + 2\pi l \frac{x}{d} \right)}{lm}.
\]

\[
= \frac{(m-1)(n-1)}{2} - \frac{1}{2} + \frac{d}{\pi} \sum_{l=1}^{\infty} \frac{\sin \left( 2\pi l \frac{x}{d} \right)}{l}.
\]

A final comment is in order. If \( (x + nk_0)/m \in \mathbb{Z} \) for some \( 0 \leq k_0 \leq m-1 \), then it is readily verified that:

1. \( x/d \in \mathbb{Z} \). Effectively, if some \( l \in \mathbb{Z} \) exists such that \( (x + nk_0)/m = l' \), then \( (x + n'dk_0)/(m'd) = l \), which implies that

\[
\frac{x}{d} = m'l - n'k_0 \in \mathbb{Z}.
\]

2. if \( k_1 \neq k_0 \) verifies that \( 0 \leq k_1 \leq m-1 \) and also \( (x + nk_1)/m = l_1 \in \mathbb{Z} \), then \( |k_2 - k_1| \) is a multiple of \( m' \). To check it, just observe that from \( (x + nk_0)/m = l_0 \) and \( (x + nk_1)/m = l_1 \) one gets \( n(k_1 - k_0) = m(l_1 - l_0) \) or \( n'(k_1 - k_0) = m'(l_1 - l_0) \); since \( \gcd(m', n') = 1 \), then \( n' \) divides \( (l_1 - l_0) \), so finally \( (k_1 - k_0) = sm' \) for some integer \( s \).

3. for each integer \( r \) such that \( k_r = k_0 + rm' \in \{0, 1, \ldots, m-1\} \), it also holds \( (x + nk_r)/m \in \mathbb{Z} \). To show it, use

\[
\frac{x + nk_0}{m} = l_0, \quad \text{and} \quad \frac{x + n'dk_0}{m'd} = l_0,
\]

to obtain

\[
\frac{x}{d} = m'l_0 - n'k_0 = m'l_0 + m'rn' - n'k_0 - m'rn' \\
= m'(l_0 + rn') - n'(k_0 + rm').
\]

These three items above show that there are exactly \( d \) distinct \( k \)'s in \( \{0, 1, \ldots, m-1\} \) for which \( (x + nk_j)/m \in \mathbb{Z} \). Thus, (4) holds true in this case too, because

\[
\sum_{k=0}^{m-1} \left[ \frac{x + nk}{m} \right] = \left( x + \frac{(m-1)n}{2} - \frac{m}{2} + \frac{d}{\pi} \sum_{l=1}^{\infty} \sum_{k=0}^{m-1} \frac{\sin \left( 2\pi ln'k + 2\pi l \frac{x}{d} \right)}{lm} \right) + \frac{d}{2}.
\]
(we have added \(d/2\) to correct the value of \(f(\cdot)\) in the \(d\) cases in which its argument is an integer). Therefore

\[
\sum_{k=0}^{m-1} \left\lfloor \frac{x + nk}{m} \right\rfloor = \frac{(m-1)(n-1)}{2} - \frac{1}{2} + \frac{d}{2} + d \left( \frac{x}{d} - \frac{1}{2} + \frac{1}{\pi} \sum_{l=1}^{\infty} \sin \left( \frac{2\pi l x}{d} \right) \frac{l}{d} + \frac{1}{2} \right)
\]

\[
= \frac{(m-1)(n-1)}{2} + \frac{d-1}{2} + d \left( f \left( \frac{x}{d} \right) + \frac{1}{2} \right)
\]

\[
= \frac{(m-1)(n-1)}{2} + \frac{d-1}{2} + d \left\lfloor \frac{x}{d} \right\rfloor,
\]

and we are done.

References


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