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A short proof of a concrete sum

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Abstract: We give an alternative proof of a formula that generalizes Hermite's identity. Instead involving modular arithmetic, our short proof relies on the Fourier-type expansion for the floor function and on a trigonometric formula.
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A closed form for $\sum_{k=0}^{m-1} \left\lfloor \frac{x+nk}{m} \right\rfloor$, where x is a real number and m, n are integers with $m > 0$ can be found in [1], where the authors use modular arithmetic to establish the formula in an elementary (although somewhat long) manner. Our aim is to give a short proof of the above-mentioned result. To this end, define the *fractional part* of any real x by $\{x\} = x - \lfloor x \rfloor$ and notice that it is a periodic piecewise linear function, discontinuous at each integer point, whose Fourier expansion gives us

$$\lfloor x \rfloor = x - \frac{1}{2} + \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{\sin(2\pi jx)}{j}, \qquad x \in \mathbb{R} \setminus \mathbb{Z}. \tag{1}$$

If $f(x)$ stands for the right-hand side of (1), then $f(n) = n - 1/2$ for each integer n, and thus $\lfloor n \rfloor = n = f(n) + 1/2$. From (1) and using that $\sum_{k=0}^{m-1} k = (m-1)m/2$, one gets

$$\sum_{k=0}^{m-1} \left\lfloor \frac{x+nk}{m} \right\rfloor = x + \frac{(m-1)n}{2} - \frac{m}{2} + \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{\sum_{k=0}^{m-1} \sin\left(2\pi j \left(\frac{x+nk}{m}\right)\right)}{j}, \tag{2}$$

provided none of the $(x+nk)/m$ is an integer.

By using $\sum_{k=0}^{p} \sin(z+ak) = \csc(a/2)\sin(a(p+1)/2)\sin(z+ap/2)$ (see [2]) we establish that

$$\sum_{k=0}^{m-1} \sin\left(2\pi j \left(\frac{x+nk}{m}\right)\right) = \frac{\sin(\pi jn)}{\sin(\pi j \frac{n}{m})} \sin(\pi jn - \pi j \frac{n}{m} + 2\pi j \frac{x}{m}). \tag{3}$$

Note that the above sum vanishes except, eventually, when the denominator at the right-hand side also vanishes. Therefore, denoting $d = \gcd(m, n)$, $m' = m/d$ and $n' = n/d$, (3) may differ from zero only when $jn/m = jn'/m' \in \mathbb{Z}$, namely, when $j = lm' = lm/d$ ($l = 1, 2, 3, \ldots$). With this in mind, (2) finally transforms to

$$\begin{aligned}
\sum_{k=0}^{m-1} \left\lfloor \frac{x+nk}{m} \right\rfloor &= x + \frac{(m-1)n}{2} - \frac{m}{2} + \frac{d}{\pi} \sum_{l=1}^{\infty} \frac{\sum_{k=0}^{m-1} \sin\left(2\pi l n' k + 2\pi l \frac{x}{d}\right)}{lm} \\
&= \frac{(m-1)(n-1)}{2} - \frac{1}{2} + \frac{d}{2} + d\left(\frac{x}{d} - \frac{1}{2} + \frac{1}{\pi}\sum_{l=1}^{\infty} \frac{\sin\left(2\pi l \frac{x}{d}\right)}{l}\right) \\
&= \frac{(m-1)(n-1)}{2} + \frac{d-1}{2} + d\left\lfloor \frac{x}{d} \right\rfloor . \qquad (4)
\end{aligned}$$

A final comment is in order. If $(x + nk_0)/m \in \mathbb{Z}$ for some $0 \le k_0 \le m-1$, then it is readily verified that:

1. $x/d \in \mathbb{Z}$. Effectively, if some $l \in \mathbb{Z}$ exists such that $(x + nk_0)/m = l$, then $(x + n'dk_0)/(m'd) = l$, which implies that
$$\frac{x}{d} = m'l - n'k_0 \in \mathbb{Z}.$$
2. if $k_1 \neq k_0$ verifies that $0 \le k_1 \le m-1$ and also $(x + nk_1)/m = l_1 \in \mathbb{Z}$, then $|k_2 - k_1|$ is a multiple of m'. To check it, just observe that from $(x+nk_0)/m = l_0$ and $(x+nk_1)/m = l_1$ one gets $n(k_1 - k_0) = m(l_1 - l_0)$ or $n'(k_1 - k_0) = m'(l_1 - l_0)$; since $\gcd(m', n') = 1$, then n' divides $(l_1 - l_0)$, so finally $(k_1 - k_0) = sm'$ for some integer s.
3. for each integer r such that $k_r = k_0 + rm' \in \{0, 1, \ldots m-1\}$, it also holds $(x+nk_r)/m \in \mathbb{Z}$. To show it, use
$$\frac{x + nk_0}{m} = l_0, \qquad \text{and} \qquad \frac{x + n'dk_0}{m'd} = l_0,$$
to obtain
$$\begin{aligned}
\frac{x}{d} &= m'l_0 - n'k_0 = m'l_0 + m'rn' - n'k_0 - m'rn' \\
&= m'(l_0 + rn') - n'(k_0 + rm').
\end{aligned}$$

These three items above show that there are exactly d distinct k_js in $\{0, 1, \ldots, m-1\}$ for which $(x + nk_j)/m \in \mathbb{Z}$. Thus, (4) holds true in this case too, because

$$\sum_{k=0}^{m-1} \left\lfloor \frac{x+nk}{m} \right\rfloor = \left(x + \frac{(m-1)n}{2} - \frac{m}{2} + \frac{d}{\pi} \sum_{l=1}^{\infty} \frac{\sum_{k=0}^{m-1} \sin\left(2\pi l n' k + 2\pi l \frac{x}{d}\right)}{lm} \right) + \frac{d}{2}$$

(we have added $d/2$ to correct the value of $f(\cdot)$ in the d cases in which its argument is an integer). Therefore

$$\begin{aligned}\sum_{k=0}^{m-1}\left\lfloor\frac{x+nk}{m}\right\rfloor &= \frac{(m-1)(n-1)}{2}-\frac{1}{2}+\frac{d}{2}+d\left(\frac{x}{d}-\frac{1}{2}+\frac{1}{\pi}\sum_{l=1}^{\infty}\frac{\sin\left(2\pi l\frac{x}{d}\right)}{l}+\frac{1}{2}\right)\\ &= \frac{(m-1)(n-1)}{2}+\frac{d-1}{2}+d\left(f\left(\frac{x}{d}\right)+\frac{1}{2}\right)\\ &= \frac{(m-1)(n-1)}{2}+\frac{d-1}{2}+d\left\lfloor\frac{x}{d}\right\rfloor,\end{aligned}$$

and we are done.

References

[1] Graham, R. L., Knuth, D. E., & Patashnik, O. (1994) *Concrete Mathematics: A Foundation for Computer Science*. Second edition. Addison-Wesley Publishing Co., Reading, Massachusetts.

[2] The Wolfram Functions Site, `http://functions.wolfram.com/ElementaryFunctions/Sin/23/01/0003/`.