Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 23, 2017, No. 3, 27–34

All associated Stirling numbers are arithmetical triangles

Khaled Ben Letaïef

Aeronautics and aerospace high graduate engineer
16 Bd du Maréchal de Lattre, apt. 095, 21300 Chenove, France
e-mail: letaiev@gmail.com

Received: 9 January 2016 **Accepted:** 31 January 2017

Abstract: Associated Stirling numbers of first and second kind are usually found in the literature in various forms of stairs depending on their order r. Yet, it is shown in this note that all of these numbers can be arranged, through a linear transformation, in the same arithmetical triangle structure as the 'Pascal's triangle'.
Keywords: Number theory, Associated Stirling numbers, Arithmetical triangle.
AMS Classification: 11B73, 05A18.

1 Introduction

First, one provides the plane with an orthonormal reference mark and trigonometrical direction. For positive integer r fixed, let $\mathcal{S}_r = \{S_r(n,k)\}_{n\geq 0, k\geq 0}$ be the set of the Stirling numbers of second kind at order r, laid out as in the literature, in lines according to the value of n and in columns according to k. It is pointed out that $S_r(n,k)$ is the number of partitions of a set of n elements in k subsets of size at least equal to r.

These numbers check the relation of recurrence:

$$S_r(n,k) = k.S_r(n-1,k) + \binom{n-1}{r-1} S_r(n-r,k-1)$$

One gives initially the following definition:

Definition 1. One will say that an integer sequence $t(i,j)$ has a structure of arithmetical triangle, or more simply is an *arithmetical triangle*, if and only if all its terms $t(i,j)$ are calculable starting from initial data and of a relation of recurrence of the type:

$$t(i,j) = g(i,j)t(i-1,j-1) + h(i,j)t(i-1,j)$$

where g et h are integer functions of discrete variables i, j.

Examples: The triangles of Pascal and Stirling numbers of both kinds are arithmetical.

We introduce, for any order r, two arithmetical triangles σ_r and π_r such as one find, for $r = 1$, the classical Stirling numbers of first and second kind. Their associates are obtained for the orders r higher than 1 but we will arrange them on the plane in an original way, i.e. in the form of arithmetical triangle.

Indeed, in the literature ([1,4–8]), these numbers are almost always represented in the form of lines, according to the value of an index n related to the combinatorial interpretation of the Stirling numbers: one then obtains, with the higher orders, structures of numbers rather heterogeneous: modular properties of the case $r = 2$ are studied by Howard [4].

There exists however an exception in the authors being interested in the enumeration of the *hierarchies* $H(n,k)$ of level k on a set with n elements or in the *phylogenetic trees*: one can find that for example at L. Comtet (1970) [1] but also at P. Hilton *et al.* [2,3]. The latter introduce what they name the *factors of Stirling*, which are precisely the associated Stirling numbers of second kind with order 2 laid out in arithmetic triangle, but they are not recognized as such. Besides, this familiar structure is not generalized at any order and for both kind of associated Stirling numbers. It is what we will try to make in this study.

2 Linear transformation of the associated Stirling numbers of first kind

First, we focus on the associated Stirling numbers of first kind at order $r \geqslant 1$, noted $d_r(n,k)$ in the literature: $d_r(n,k)$ is defined as the number of permutations of a set of n elements having exactly k cycles, all length $\geqslant r$. These numbers check the relation of following recurrence [7]:

Relation 1.

$$d_r(n,k) = (n-1)d_r(n-1,k) + (n-1)_{r-1}d_r(n-r,k-1)$$

where $(n)_k$ indicates the downward factorial, as well as the initial conditions:

$$\begin{aligned} d_r(n,k) &= 0 \qquad \forall n \leqslant kr-1 \\ d_r(n,1) &= (n-1)! \end{aligned}$$

Example for $r = 3$: [8]

2			
6			
24			
120	40		
720	420		
5040	3948		
40320	38304	2240	
362880	396576	50400	
3628800	4419360	859320	
39916800	53048160	13665960	246400

One sees in the general case that these tables are not arithmetical triangles but stair steps whose size increases with r. To be an arithmetical triangle, it would be necessary at least that each line contains a number of elements equal to its subscript. The following theorem then gathers in only one structure, for any order, the various representations of the $d_r(n,k)$:

Theorem 1. *For all $r \geqslant 1$, the $\{d_r(n,k)\}_{n\geq 1, k\geq 1}$ has a structure of arithmetical triangle: the latter will be noted σ_r and its terms $\{D_r(m,k)\}_{m\geq 1, k\geq 1}$.*

One understands thereby that it is possible, for all r, to lay out the $d_r(n,k)$ in triangular structure in the plane, in such a way that they follow a standard law of recurrence in the same way as in Definition 1.

Proof. Let $r \geqslant 1$ be fixed and let R be the application in the space of numerical sequences s with two integer variables n and k defined by:

$$R : s(n,k) \mapsto R(s(n,k)) = s(n+k-1,k)$$

Or:

$$s(n,k) = R(s(n-(k-1),k))$$

This application is linear. Indeed, for all real numbers a, b, all sequences u, v, and all natural integers n and k:

$$\begin{aligned}
&[R(a.u+b.v)](n,k) = R[(a.u+b.v)(n,k)] \\
&= (a.u+b.v)(n+k-1,k) = a.u(n+k-1,k) + b.v(n+k-1,k) \\
&= a.R(u(n,k)) + b.R(v(n,k)) = [a.R(u) + b.R(v)](n,k)
\end{aligned}$$

Hence:

$$R(a.u+b.v) = a.R(u) + b.R(v)$$

One already see that, for $k=1$, the $s(n,1)$ are invariants of R given that:

$$s(n,1) = R(s(n,1))$$

Let us take here $s \equiv d_r$. By definition, the d_r verify the previous relation 1 :

$$d_r(n,k) = (n-1)d_r(n-1,k) + (n-1)_{r-1}d_r(n-r,k-1)$$

Using this expression as a function of the transformation R, we obtain:

$$\begin{aligned} R(d_r(n-(k-1),k)) &= (n-1)R(d_r(n-1-(k-1),k)) \\ &+(n-1)_{r-1}R(d_r(n-r-(k-2),k-1)) \end{aligned}$$

Let us prove now that, $\forall i \geqslant 1$, the application of R^i (i.e. i times the composition of R with itself) to Relation 1 gives us the following:

Relation 2.

$$\begin{aligned} R^i(d_r(n-i(k-1),k)) &= (n-1)R^i(d_r(n-1-i(k-1),k)) \\ &+(n-1)_{r-1}R^i(d_r(n-r-i(k-2),k-1)) \end{aligned}$$

- The case $i = 1$ has just been proved.
- Let us suppose it true for one fixed i.

It is enough to express each term in R^i of Relation 2 as a function of R^{i+1}, namely:

$$\begin{aligned} R^i(d_r(n-i(k-1),k)) &= R^{i+1}(d_r(n-(i+1)(k-1),k)) \\ R^i(d_r(n-1-i(k-1),k)) &= R^{i+1}(d_r(n-1-(i+1)(k-1),k)) \\ R^i(d_r(n-r-i(k-2),k-1)) &= R^{i+1}(d_r(n-r-(i+1)(k-2),k-1)) \end{aligned}$$

This proves the heredity of our relation to the rank $i+1$. Now, it is enough to set down $i = r-1$ in relation 1, which gives:

$$\begin{aligned} R^{r-1}(d_r(n-(r-1)(k-1),k)) &= (n-1)R^{r-1}(d_r(n-(r-1)(k-1)-1,k))+ \\ &(n-1)_{r-1}R^{r-1}(d_r(n-(r-1)(k-2)-r,k-1)) \end{aligned}$$

hence:

$$\begin{aligned} R^{r-1}(d_r(n-(r-1)(k-1),k)) &= (n-1)R^{r-1}(d_r(n-(r-1)(k-1)-1,k))+ \\ &(n-1)_{r-1}R^{r-1}(d_r(n-(r-1)(k-1)-1,k-1)) \end{aligned}$$

After having set down $m = n-(r-1)(k-1)$, this new relation becomes:

$$\begin{aligned} R^{r-1}(d_r)(m,k) &= (m+(r-1)(k-1)-1)R^{r-1}(d_r)(m-1,k)+ \\ &(m+(r-1)(k-1)-1)_{r-1}R^{r-1}(d_r)(m-1,k-1) \end{aligned}$$

thus:

$$\begin{aligned} D_r(m,k) &= (m+(r-1)(k-1)-1)D_r(m-1,k)+ \\ &(m+(r-1)(k-1)-1)_{r-1}D_r(m-1,k-1) \end{aligned}$$

with

$$D_r = R^{r-1}(d_r)$$

One finds well with the new sequence $D_r(m,k)$ an arithmetical triangle-type relation like in Definition 1. □

The application of R^{r-1} to the associated Stirling numbers of first kind at order r thus gives them a structure of arithmetical triangle. We now will see that there exists a geometrical manner to interpret the action of R on these numbers in the plane.

Let s be an element of the space of the almost zero sequences with two natural integer variables n and k. One can represent s in a table $\mathcal{T}$ of abscissa $n \geqslant 1$ and ordinate $k \geqslant 1$, by positioning each term $s(n,k)$ in the box of coordinates (n,k).

Let n_0 be fixed. Let us name L_{n_0} the line of abscissa n_0 in table $\mathcal{T}$: it thus contains all $\{S(n_0,k)\}_k$. R transforms this line into $\{S(n_0-k,k)\}_k$: thus, any element of coordinates (n_0,k) is found with the coordinates (n_0-k+1,k), which corresponds geometrically to a plane rotation of L_{n_0} of angle $\frac{\pi}{4}$ around the box of indices $(n_0,1)$: L_{n_0} changes thus in diagonal of a square with side L_{n_0}.

In a more general way, let be a "line" $\mathcal{D}_{n_0,p_0}$ of elements $s(n,k)$ starting from $s(n_0,1)$ (invariant of R) and passing through $s(p_0,2)$ with $p_0 \geqslant 1$. Geometrically, the subscripts of its elements (n,k) check the following discrete equation:

$$(k-1)(n_0-p_0)+(n-n_0)=0 \qquad \forall n,k \geqslant 1$$

Let us apply R to $\mathcal{D}_{n_0,p_0}$:

$$\begin{aligned} R\left(\mathcal{D}_{n_0,p_0}\right) &= R\left[s(n,k)_{(n,k)\in\mathcal{D}_{n_0,p_0}}\right] \\ &= s(n-(k-1),k)_{(n,k)\in\mathcal{D}_{n_0,p_0}} \end{aligned}$$

Yet, the equation of $\mathcal{D}_{n_0,p_0}$ can be also written:

$$\begin{aligned} (k-1)(n_0-p_0)+(n-n_0) &= (k-1)(n_0-p_0)+(n-(k-1)-n_0)+(k-1) \\ &= (k-1)(n_0-(p_0-1))+(n-n_0)=0 \end{aligned}$$

Thus R transforms the line $\mathcal{D}_{n_0,p_0}$ into the new line $\mathcal{D}_{n_0,p_0-1}$ of equation:

$$(k-1)(n_0-(p_0-1))+(n-n_0)=0$$

passing through this time by the points: $s(n_0,1)$ and $s(p_0-1,2)$.

The transformation R is equivalent to a plane "rotation" of $\mathcal{D}_{n_0,p_0}$ of centre $s(n_0,1)$ and "angle" $(\mathcal{D}_{n_0,p_0},\mathcal{D}_{n_0,p_0-1})$. To be more rigorous, it is the real geometrical line confused with $\mathcal{D}_{n_0,p_0}$ which undergoes a rotation: the "rotation" of the elements $s(n_0,p_0)$ of $\mathcal{D}_{n_0,p_0}$, as for it, is rather their projection parallel to the axis of the n on $\mathcal{D}_{n_0,p_0}$.

Definition 2. The transformation R applied to the definite sequences will be known as *geometrical rotation* or simply *rotation*.

The traditional concepts of angle and centre of rotation could also be used. Moreover, one will call *centre of rotation associated with* m the R-invariant term of the line associated to m.

NB: One can already notice that the application of R^p to a "line" of zero slope gives a new slope equal to p.

Finally, R may be naturally prolonged into an homeomorphism of the plane: we can then consider it as an *elastic transformation*.

Let us apply now this result to the associated Stirling numbers of first kind with order $r = 3$, namely $d_3(n, k)$, in the form which one traditionally finds in the literature, noted Σ_0:

2			
6			
24			
120	40		
720	420		
5040	3948		
40320	38304	2240	
362880	396576	50400	
3628800	4419360	859320	
39916800	53048160	13665960	246400
...	...	...	...

Hence:

2				
6	40			
24	420	2240		
120	3948	50400	246400	
...	...	...	...	...

One finds an application of Theorem 1, according to which Σ_0 will become arithmetical at the end of $r - 1 = 2$ iterations of R.

3 Linear transformation of the associated Stirling numbers of second kind

At order r, these numbers check the law of following recurrence [1]:

Relation 3.

$$s_r(n, k) = ks_r(n - 1, k) + \binom{n-1}{r-1} s_r(n - r, k - 1)$$

We can see that the terms in s_r in this relation are of the same form as the d_r in Relation 1. It is then possible to define $r - 1$ linear transformations of the same type as R and to extend Theorem 1 to the numbers $s_r(n, k)$:

Theorem 2. *For all $r \geqslant 1$, the $\{s_r(n,k)\}_{\substack{n \geq 1 \\ k \geq 1}}$ has a structure of arithmetical triangle.*

One will note π_r this structure and $\{S_r(m,k)\}_{\substack{m \geq 1 \\ k \geq 1}}$ its terms.

Proof. Like above one could show that:

$$R^{r-1}(s_r)(m,k) = k.R^{r-1}(s_r)(m-1,k) + \binom{m+(r-1)(k-1)-1}{r-1} R^{r-1}(s_r)(m-1,k-1)$$

thus

$$S_r(m,k) = k.S_r(m-1,k) + \binom{m+(r-1)(k-1)-1}{r-1} S_r(m-1,k-1)$$

with:

$$S_r = R^{r-1}(s_r)$$

□

Let us pose the following definitions:

Definition 3. One calls *line associated with a natural integer* n in π_r or σ_r, the whole of the terms corresponding to $s_r(n,k)$ or $d_r(n,k)$ for all k.

Definition 4. For all positive integer r and all integer h, one will call respectively Σ_r^h and Π_r^h the arithmetical structures obtained after h iterations of R applied to the arithmetical triangles σ_r and π_r.

Thus, for $h = 0$, the structures Σ_r^0 and Π_r^0 are σ_r and π_r, respectively. Moreover, according to Theorems 1 and 2, for $h = -(r-1) = 1-r$, Σ_r^{1-r} and Π_r^{1-r} are precisely the structures of numbers $d_r(n,k)$ and $S_r(n,k)$ laid out such as we find them in the literature.

4 Conclusion

We propose to apply a geometrical transformation to the associated Stirling numbers of both kind and any order: it makes them move from the "rough" and various state in which they are traditionally found in literature to the unique "compact" structure of an arithmetical triangle. This approach can lead to new striking combinatorial properties of associated Stirling numbers. This will be the subject of a forthcoming work.

Acknowledgements

The author would like to warmly thank Prof. K. Atanassov, the NNTDM Journal, as well as the anonymous referees, for their genial reception of this discrete article and their attentive evaluation work.

References

[1] Comtet, L. (1970) *Analyse combinatoire*, PUF.

[2] Hilton, P. & Pedersen, J. (1999) Two recurrence relations for Stirling factors, *Bulletin Belg. Math. Soc. Simon Stevin*, 6, 615–623.

[3] Hilton, P., et al. (1994) On partitions, surjections and Stirling numbers, *Bulletin Belg. Math. Soc. Simon Stevin*, 1, 713–725.

[4] Howard, F.T. (1990), Congruences for the Stirling numbers and associated Stirling numbers, *Acta Arithmetica*, LV, 29–41.

[5] Sandor, J., & Crstici, B. (2004) *Handbook of number theory II*, Kluwer Acad. Publ., Chapter 5, 459–618.

[6] Weisstein, Eric W., "Stirling Number of the Second Kind." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/StirlingNumberoftheSecondKind.html`

[7] Weisstein, Eric W., "Stirling Number of the First Kind." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/StirlingNumberoftheFirstKind.html`

[8] Weisstein, Eric W., "Permutation Cycle." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/PermutationCycle.html`