Notes on Number Theory and Discrete Mathematics
Print ISSN 1310-5132, Online ISSN 2367-8275
Vol. 23, 2017, No. 2, 104–116

Closed-form evaluations of Fibonacci–Lucas reciprocal sums with three factors

Robert Frontczak *

Landesbank Baden-Wuerttemberg
Am Hauptbahnhof 2, 70173 Stuttgart, Germany
e-mail: robert.frontczak@lbbw.de

Received: 3 October 2016 **Accepted:** 31 March 2017

Abstract: In this article we present expressions for certain types of reciprocal Fibonacci and Lucas sums. The common feature of the sums is that in each case the denominator of the summand consists of a product of three Fibonacci or Lucas numbers.
Keywords: Fibonacci number, Lucas number, Reciprocal sum.
AMS Classification: 11B37, 11B39.

1 Introduction

Generalized Fibonacci numbers G_n may be defined through the second order recurrence relation $G_{n+1} = G_n + G_{n-1}$, where the initial terms G_0 and G_1 need to be specified. In this paper we focus on the two most popular members of this family: The Fibonacci numbers F_n and Lucas numbers L_n, which are defined by initial conditions $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$, respectively. The Binet forms are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n, \qquad n \geq 0, \tag{1.1}$$

where α and β are roots of the quadratic equation $x^2 - x - 1 = 0$, i.e., $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

*Disclaimer: Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

The goal of this study is to evaluate special finite and infinite reciprocal sums of these numbers. The interest is not new and some open problems still exist. For instance, no simple expressions for the sums

$$\sum_{i=1}^{\infty}\frac{1}{F_i},\quad \sum_{i=1}^{\infty}\frac{1}{L_i},\quad \sum_{i=1}^{\infty}\frac{(-1)^i}{F_i}\ \text{and}\ \sum_{i=1}^{\infty}\frac{(-1)^i}{L_i}$$

are known. The summation of reciprocals of generalized Fibonacci numbers is a challenging issue. It is discussed in several articles where different approaches are applied. The main lines of research are: establishing algebraic relationships (i.e. reduction formulas) for these sums, expressing the sums in terms of special functions and a direct evaluation. Reduction formulas may be found in the articles [5], [6], [7], [16], [17] or [23]. Expressions involving Elliptic functions, the Lambert series and/or Theta functions are derived in [1], [2], [4], [11], [12] or [13].

A direct evaluation of reciprocal Fibonacci sums is also possible in many cases. Consequently, many articles covering the topic exist. Two classical results are contained in the articles [3] and [10], where special families of sums are evaluated exactly. Focusing on reciprocal sums with three and more factors we refer to the research work of Melham. In a series of papers starting in 2000 he gives closed forms for many types of these sums (see [14], [15], [19], [20], [21]). However, as far as we can say, the sums that we study here have not been considered by the author.

The derivation of the results contained in the present study is based on the following results, which we state as theorems.

Theorem 1.1. *Let $m \geq 1$ and $n \geq 0$ be integers. Then*

$$\sum_{i=1}^{N}\frac{(-1)^{m(i+1)}}{F_{mi+n}F_{m(i+1)+n}} = \frac{F_{mN}}{F_mF_{m+n}F_{m(N+1)+n}} \tag{1.2}$$

and

$$\sum_{i=1}^{N}\frac{(-1)^{m(i+1)}}{L_{mi+n}L_{m(i+1)+n}} = \frac{F_{mN}}{F_mL_{m+n}L_{m(N+1)+n}}. \tag{1.3}$$

Especially,

$$\sum_{i=1}^{\infty}\frac{(-1)^{m(i+1)}}{F_{mi+n}F_{m(i+1)+n}} = \frac{1}{F_mF_{m+n}\alpha^{m+n}} \tag{1.4}$$

and

$$\sum_{i=1}^{\infty}\frac{(-1)^{m(i+1)}}{L_{mi+n}L_{m(i+1)+n}} = \frac{1}{\sqrt{5}F_mL_{m+n}\alpha^{m+n}}. \tag{1.5}$$

These expressions are a consequence of the main result in [12]. Other proofs can be found in [9] and [19]. The special case where m is even also appears in [18]. The subcase m even and $n = 0$ is presented in [22].

The second theorem that is relevant for our study contains findings from [8]:

Theorem 1.2. *Let $m, n \geq 1$ be integers. Then*

$$\sum_{i=1}^{N}\frac{(-1)^{m(i+1)}}{F_{m(i-1)+n}F_{m(i+1)+n}} = \frac{1}{F_nF_{2m}}\Big(\frac{F_{m(N+1)}}{F_{m(N+1)+n}} + \frac{F_{mN}}{F_{mN+n}} - \frac{F_m}{F_{m+n}}\Big), \tag{1.6}$$

$$\sum_{i=1}^{N}\frac{(-1)^{m(i+1)}}{L_{m(i-1)+n}L_{m(i+1)+n}}=\frac{1}{5F_nF_{2m}}\Big(\frac{2}{L_n}+\frac{L_m}{L_{m+n}}-\frac{L_{mN}}{L_{mN+n}}-\frac{L_{m(N+1)}}{L_{m(N+1)+n}}\Big),\tag{1.7}$$

and

$$\sum_{i=1}^{N}\frac{(-1)^{m(i+1)}}{L_{m(i-1)}L_{m(i+1)}}=\frac{1}{2F_{2m}}\Big(\frac{F_{m(N+1)}}{L_{m(N+1)}}+\frac{F_{mN}}{L_{mN}}-\frac{F_m}{L_m}\Big).\tag{1.8}$$

Especially,

$$\sum_{i=1}^{\infty}\frac{(-1)^{m(i+1)}}{F_{m(i-1)+n}F_{m(i+1)+n}}=\frac{1}{F_nF_{2m}}\Big(\frac{2}{\alpha^n}-\frac{F_m}{F_{m+n}}\Big),\tag{1.9}$$

$$\sum_{i=1}^{\infty}\frac{(-1)^{m(i+1)}}{L_{m(i-1)+n}L_{m(i+1)+n}}=\frac{1}{5F_nF_{2m}}\Big(\frac{2}{L_n}+\frac{L_m}{L_{m+n}}-\frac{2}{\alpha^n}\Big),\tag{1.10}$$

and

$$\sum_{i=1}^{\infty}\frac{(-1)^{m(i+1)}}{L_{m(i-1)}L_{m(i+1)}}=\frac{\sqrt{5}}{5F_{2m}}-\frac{1}{2L_m^2}.\tag{1.11}$$

Equation (1.8) may be seen as a special case of (1.7) for $n=0$.
The formulas (1.2)–(1.11) will be treated as basic forms. These forms may be combined to produce more results of this nature. Specifically, some combinations produce exact evaluations of reciprocal sums with three factors. In the next section we present applications of the idea.

2 Results

We require the following identities:

Lemma 2.1. *Let u and v be integers. Then*

$$F_{u-v}=(-1)^v(F_uF_{v+1}-F_{u+1}F_v)\tag{2.1}$$

and

$$L_{u-v}=(-1)^v(L_uF_{v+1}-L_{u+1}F_v).\tag{2.2}$$

If $u+v$ and $u-v$ have equal parity then

$$F_{u+v}-F_{u-v}=\begin{cases}L_uF_v & \text{if } v \text{ is even}\\ F_uL_v & \text{if } v \text{ is odd,}\end{cases}\tag{2.3}$$

and

$$L_{u+v}-L_{u-v}=\begin{cases}5F_uF_v & \text{if } v \text{ is even}\\ L_uL_v & \text{if } v \text{ is odd.}\end{cases}\tag{2.4}$$

Proof. The first two equations can be proved using the relations

$$F_{a+b}=F_aF_{b+1}+F_{a-1}F_b,$$

and

$$L_{a+b}=L_aF_{b+1}+F_bL_{a-1},$$

together with $F_{-n}=(-1)^{n+1}F_n$ (equation (2.1) is given in [18]). The last two statements follow from the Binet forms. We omit the details. □

Now, we state the main results of our investigation. Each finding is presented as a separate proposition.

Proposition 2.2. *Let $m, n \geq 1$ be any integers. Define the sums $T_1(m,n,N)$ and $T_2(m,n,N)$ as*

$$T_1(m,n,N) = \sum_{i=1}^{N} (-1)^{m(i+1)} \frac{F_{mi+n+1}}{F_{m(i-1)+n} F_{mi+n} F_{m(i+1)+n}}, \tag{2.5}$$

and

$$T_2(m,n,N) = \sum_{i=1}^{N} (-1)^{m(i+1)} \frac{F_{mi+n-1}}{F_{m(i-1)+n} F_{mi+n} F_{m(i+1)+n}}. \tag{2.6}$$

Then

$$T_1(m,n,N) = \begin{cases} \frac{F_{m+1}}{F_n F_m F_{2m}} \left(\frac{F_{m(N+1)}}{F_{m(N+1)+n}} + \frac{F_{mN}}{F_{mN+n}} - \frac{F_m}{F_{m+n}} \right) - \frac{F_{mN}}{F_m^2 F_{m+n} F_{m(N+1)+n}}, & m \text{ even} \\ \frac{F_{m+1}}{F_n F_m F_{2m}} \left(\frac{F_{m(N+1)}}{F_{m(N+1)+n}} + \frac{F_{mN}}{F_{mN+n}} - \frac{F_m}{F_{m+n}} \right) + \frac{F_{mN}}{F_m^2 F_{m+n} F_{m(N+1)+n}}, & m \text{ odd.} \end{cases} \tag{2.7}$$

and

$$T_2(m,n,N) = \begin{cases} \frac{F_{m-1}}{F_n F_m F_{2m}} \left(\frac{F_{m(N+1)}}{F_{m(N+1)+n}} + \frac{F_{mN}}{F_{mN+n}} - \frac{F_m}{F_{m+n}} \right) - \frac{F_{mN}}{F_m^2 F_{m+n} F_{m(N+1)+n}}, & m \text{ even} \\ \frac{F_{m-1}}{F_n F_m F_{2m}} \left(\frac{F_{m(N+1)}}{F_{m(N+1)+n}} + \frac{F_{mN}}{F_{mN+n}} - \frac{F_m}{F_{m+n}} \right) + \frac{F_{mN}}{F_m^2 F_{m+n} F_{m(N+1)+n}}, & m \text{ odd} \end{cases} \tag{2.8}$$

Proof. We combine equations (1.2) and (1.6) and calculate

$$\sum_{i=1}^{N} (-1)^{m(i+1)} \frac{F_{mi+n} - F_{m(i-1)+n}}{F_{m(i-1)+n} F_{mi+n} F_{m(i+1)+n}}. \tag{2.9}$$

From Lemma 2.1 we have

$$F_{mi+n} - F_{m(i-1)+n} = \begin{cases} F_{mi+n}(1 - F_{m+1}) + F_m F_{mi+n+1}, & m \text{ even} \\ F_{mi+n}(1 + F_{m+1}) - F_m F_{mi+n+1}, & m \text{ odd.} \end{cases} \tag{2.10}$$

Inserting this relation into the above sum, simplifying and using the RHS of (1.2) and (1.6), respectively, proves the first statement. To establish the second formula, observe that by definition of the Fibonacci numbers we can write

$$F_{mi+n} - F_{m(i-1)+n} = \begin{cases} F_{mi+n}(1 - F_{m-1}) + F_m F_{mi+n-1}, & m \text{ even} \\ F_{mi+n}(1 + F_{m-1}) - F_m F_{mi+n-1}, & m \text{ odd.} \end{cases} \tag{2.11}$$

□

Corollary 2.3.

$$T_1(m,n,\infty) = \begin{cases} \frac{F_{m+1}}{F_n F_m F_{2m}} \left(\frac{2}{\alpha^n} - \frac{F_m}{F_{m+n}} \right) - \frac{1}{F_m^2 F_{m+n} \alpha^{m+n}}, & m \text{ even} \\ \frac{F_{m+1}}{F_n F_m F_{2m}} \left(\frac{2}{\alpha^n} - \frac{F_m}{F_{m+n}} \right) + \frac{1}{F_m^2 F_{m+n} \alpha^{m+n}}, & m \text{ odd,} \end{cases} \tag{2.12}$$

and

$$T_2(m,n,\infty) = \begin{cases} \frac{F_{m-1}}{F_n F_m F_{2m}} \left(\frac{2}{\alpha^n} - \frac{F_m}{F_{m+n}} \right) - \frac{1}{F_m^2 F_{m+n} \alpha^{m+n}}, & m \text{ even} \\ \frac{F_{m-1}}{F_n F_m F_{2m}} \left(\frac{2}{\alpha^n} - \frac{F_m}{F_{m+n}} \right) + \frac{1}{F_m^2 F_{m+n} \alpha^{m+n}}, & m \text{ odd,} \end{cases} \tag{2.13}$$

Proposition 2.4. *Let $n \geq 1$ be an integer. Let further m be even such that $m/2$ is even, i.e. $4|m$. Define*

$$T_3(m,n,N) = \sum_{i=1}^{N} \frac{L_{mi-m/2+n}}{F_{m(i-1)+n}F_{mi+n}F_{m(i+1)+n}}. \tag{2.14}$$

Similarly, for m even with $m/2$ odd define

$$T_4(m,n,N) = \sum_{i=1}^{N} \frac{F_{mi-m/2+n}}{F_{m(i-1)+n}F_{mi+n}F_{m(i+1)+n}}. \tag{2.15}$$

Then

$$T_3(m,n,N) = \frac{1}{F_{m/2}}\Big(\frac{1}{F_nF_{2m}}\Big(\frac{F_{m(N+1)}}{F_{m(N+1)+n}}+\frac{F_{mN}}{F_{mN+n}}-\frac{F_m}{F_{m+n}}\Big)-\frac{F_{mN}}{F_mF_{m+n}F_{m(N+1)+n}}\Big), \tag{2.16}$$

and

$$T_4(m,n,N) = \frac{1}{L_{m/2}}\Big(\frac{1}{F_nF_{2m}}\Big(\frac{F_{m(N+1)}}{F_{m(N+1)+n}}+\frac{F_{mN}}{F_{mN+n}}-\frac{F_m}{F_{m+n}}\Big)-\frac{F_{mN}}{F_mF_{m+n}F_{m(N+1)+n}}\Big). \tag{2.17}$$

Proof. The statements also follow from Lemma 2.1:

$$F_{mi+n} - F_{m(i-1)+n} = \begin{cases} L_{mi+n-m/2}F_{m/2} & \text{if } \frac{m}{2} \text{ is even} \\ F_{mi+n-m/2}L_{m/2} & \text{if } \frac{m}{2} \text{ is odd.} \end{cases} \tag{2.18}$$

□

Corollary 2.5. *With m and n from above, we have*

$$T_3(m,n,\infty) = \frac{1}{F_{m/2}}\Big(\frac{1}{F_nF_{2m}}\Big(\frac{2}{\alpha^n}-\frac{F_m}{F_{m+n}}\Big)-\frac{1}{F_mF_{m+n}\alpha^{m+n}}\Big), \tag{2.19}$$

and

$$T_4(m,n,\infty) = \frac{1}{L_{m/2}}\Big(\frac{1}{F_nF_{2m}}\Big(\frac{2}{\alpha^n}-\frac{F_m}{F_{m+n}}\Big)-\frac{1}{F_mF_{m+n}\alpha^{m+n}}\Big). \tag{2.20}$$

Remark 2.6. *It is obvious that $T_1(1,n,N) = T_2(1,n-1,N)$ and $T_4(2,n,N) = T_2(2,n,N)$. Also, since $L_{4i-2+n} = F_{4i+n} - F_{4(i-1)+n}$ the expression for $T_3(4,n,N)$ equals a simple difference of the two basic forms, too.*

To illustrate the results obtained so far, we give explicit examples:

$$T_1(2,1,N) = \sum_{i=1}^{N} \frac{F_{2i+2}}{F_{2i-1}F_{2i+1}F_{2i+3}} = \frac{2}{3}\Big(\frac{F_{2N+2}}{F_{2N+3}}+\frac{F_{2N}}{F_{2N+1}}\Big)-\frac{1}{2}\frac{F_{2N}}{F_{2N+3}}-\frac{1}{3}, \tag{2.21}$$

$$T_1(2,1,\infty) = \frac{1}{3}+\frac{1}{6\alpha^3}, \tag{2.22}$$

$$T_1(3,3,N) = \sum_{i=1}^{N} (-1)^{i+1}\frac{F_{3i+4}}{F_{3i}F_{3i+3}F_{3i+6}} = \frac{3}{32}\Big(\frac{F_{3N+3}}{F_{3N+6}}+\frac{F_{3N}}{F_{3N+3}}\Big)+\frac{1}{32}\frac{F_{3N}}{F_{3N+6}}-\frac{3}{128}, \tag{2.23}$$

$$T_1(3,3,\infty) = \frac{3}{128} - \frac{1}{64\alpha^6}, \tag{2.24}$$

$$T_2(2,2,N) = \sum_{i=1}^{N} \frac{F_{2i+1}}{F_{2i}F_{2i+2}F_{2i+4}} = \frac{1}{3}\Big(\frac{F_{2N+2}}{F_{2N+4}} + \frac{F_{2N}}{F_{2N+2}}\Big) - \frac{1}{3}\frac{F_{2N}}{F_{2N+4}} - \frac{1}{9}, \tag{2.25}$$

$$T_2(2,2,\infty) = \frac{1}{9} - \frac{1}{9\alpha^4}, \tag{2.26}$$

$$T_2(3,1,N) = \sum_{i=1}^{N} (-1)^{i+1} \frac{F_{3i}}{F_{3i-2}F_{3i+1}F_{3i+4}} = \frac{1}{16}\Big(\frac{F_{3N+3}}{F_{3N+4}} + \frac{F_{3N}}{F_{3N+1}}\Big) + \frac{1}{12}\frac{F_{3N}}{F_{3N+4}} - \frac{1}{24}, \tag{2.27}$$

and

$$T_2(3,1,\infty) = \frac{1}{8\alpha^2}, \tag{2.28}$$

The next propositions contain analogous results for reciprocal Lucas sums.

Proposition 2.7. *Let $m, n \geq 1$ be any integers. Define the sums $T_5(m,n,N)$ and $T_6(m,n,N)$ as*

$$T_5(m,n,N) = \sum_{i=1}^{N} (-1)^{m(i+1)} \frac{L_{mi+n+1}}{L_{m(i-1)+n}L_{mi+n}L_{m(i+1)+n}}, \tag{2.29}$$

and

$$T_6(m,n,N) = \sum_{i=1}^{N} (-1)^{m(i+1)} \frac{L_{mi+n-1}}{L_{m(i-1)+n}L_{mi+n}L_{m(i+1)+n}}. \tag{2.30}$$

Then

$$T_5(m,n,N) = \begin{cases} \frac{F_{m+1}}{5F_nF_mF_{2m}}\Big(\frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{L_{mN}}{L_{mN+n}} - \frac{L_{m(N+1)}}{L_{m(N+1)+n}}\Big) - \frac{F_{mN}}{F_m^2 L_{m+n}L_{m(N+1)+n}}, & m \text{ even} \\ \frac{F_{m+1}}{5F_nF_mF_{2m}}\Big(\frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{L_{mN}}{L_{mN+n}} - \frac{L_{m(N+1)}}{L_{m(N+1)+n}}\Big) + \frac{F_{mN}}{F_m^2 L_{m+n}L_{m(N+1)+n}}, & m \text{ odd.} \end{cases} \tag{2.31}$$

and

$$T_6(m,n,N) = \begin{cases} \frac{F_{m-1}}{5F_nF_mF_{2m}}\Big(\frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{L_{mN}}{L_{mN+n}} - \frac{L_{m(N+1)}}{L_{m(N+1)+n}}\Big) - \frac{F_{mN}}{F_m^2 L_{m+n}L_{m(N+1)+n}}, & m \text{ even} \\ \frac{F_{m-1}}{5F_nF_mF_{2m}}\Big(\frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{L_{mN}}{L_{mN+n}} - \frac{L_{m(N+1)}}{L_{m(N+1)+n}}\Big) + \frac{F_{mN}}{F_m^2 L_{m+n}L_{m(N+1)+n}}, & m \text{ odd.} \end{cases} \tag{2.32}$$

Proof. Combine equations (1.3) and (1.7). From equation (2.2) of Lemma 2.1 we have

$$L_{mi+n} - L_{m(i-1)+n} = \begin{cases} L_{mi+n}(1 - F_{m+1}) + F_m L_{mi+n+1}, & m \text{ even} \\ L_{mi+n}(1 + F_{m+1}) - F_m L_{mi+n+1}, & m \text{ odd.} \end{cases} \tag{2.33}$$

Inserting this relation into the above sum, simplifying and using the RHS of (1.3) and (1.7), respectively, proves the first formula. The second formula is proved similarly. □

Corollary 2.8.

$$T_5(m,n,\infty) = \begin{cases} \frac{F_{m+1}}{5F_nF_mF_{2m}}\Big(\frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{2}{\alpha^n}\Big) - \frac{1}{\sqrt{5}F_m^2 L_{m+n}\alpha^{m+n}}, & m \text{ even} \\ \frac{F_{m+1}}{5F_nF_mF_{2m}}\Big(\frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{2}{\alpha^n}\Big) + \frac{1}{\sqrt{5}F_m^2 L_{m+n}\alpha^{m+n}}, & m \text{ odd} \end{cases} \tag{2.34}$$

and

$$T_6(m,n,\infty)=\begin{cases}\frac{F_{m-1}}{5F_nF_mF_{2m}}\left(\frac{2}{L_n}+\frac{L_m}{L_{m+n}}-\frac{2}{\alpha^n}\right)-\frac{1}{\sqrt{5}F_m^2L_{m+n}\alpha^{m+n}}, & m \text{ even}\\ \frac{F_{m-1}}{5F_nF_mF_{2m}}\left(\frac{2}{L_n}+\frac{L_m}{L_{m+n}}-\frac{2}{\alpha^n}\right)+\frac{1}{\sqrt{5}F_m^2L_{m+n}\alpha^{m+n}}, & m \text{ odd.}\end{cases} \quad (2.35)$$

Proposition 2.9. *Let $n \geq 1$ be an integer. Let further m be even such that $m/2$ is even, i.e. $4|m$. Define*

$$T_7(m,n,N)=\sum_{i=1}^{N}\frac{F_{mi-m/2+n}}{L_{m(i-1)+n}L_{mi+n}L_{m(i+1)+n}}. \quad (2.36)$$

Similarly, for m even with $m/2$ odd define

$$T_8(m,n,N)=\sum_{i=1}^{N}\frac{L_{mi-m/2+n}}{L_{m(i-1)+n}L_{mi+n}L_{m(i+1)+n}}. \quad (2.37)$$

Then

$$T_7(m,n,N)=\frac{1}{25F_nF_{m/2}F_{2m}}\left(\frac{2}{L_n}+\frac{L_m}{L_{m+n}}-\frac{L_{mN}}{L_{mN+n}}-\frac{L_{m(N+1)}}{L_{m(N+1)+n}}\right)-\frac{F_{mN}}{5F_{m/2}F_mL_{m+n}L_{m(N+1)+n}}, \quad (2.38)$$

and

$$T_8(m,n,N)=\frac{1}{5F_nF_{2m}L_{m/2}}\left(\frac{2}{L_n}+\frac{L_m}{L_{m+n}}-\frac{L_{mN}}{L_{mN+n}}-\frac{L_{m(N+1)}}{L_{m(N+1)+n}}\right)-\frac{F_{mN}}{F_mL_{m/2}L_{m+n}L_{m(N+1)+n}}. \quad (2.39)$$

Proof. The statement is a direct consequence of equation (2.4) of Lemma 2.1. □

Corollary 2.10.

$$T_7(m,n,\infty)=\frac{1}{25F_nF_{m/2}F_{2m}}\left(\frac{2}{L_n}+\frac{L_m}{L_{m+n}}-\frac{2}{\alpha^n}\right)-\frac{1}{5\sqrt{5}F_{m/2}F_mL_{2m}\alpha^{m+n}}, \quad (2.40)$$

and

$$T_8(m,n,\infty)=\frac{1}{5F_nF_{2m}L_{m/2}}\left(\frac{2}{L_n}+\frac{L_m}{L_{m+n}}-\frac{2}{\alpha^n}\right)-\frac{1}{\sqrt{5}F_mL_{m/2}L_{2m}\alpha^{m+n}}. \quad (2.41)$$

Remark 2.11. *It is obvious that $T_8(2,n,N)=T_6(2,n,N)$.*

Finally, we state the results in case $n=0$:

Proposition 2.12. *Let $m \geq 1$ be an integer. Define the sums $T_9(m,N)$ and $T_{10}(m,N)$ as*

$$T_9(m,N)=\sum_{i=1}^{N}(-1)^{m(i+1)}\frac{L_{mi+1}}{L_{m(i-1)}L_{mi}L_{m(i+1)}}, \quad (2.42)$$

and

$$T_{10}(m,N)=\sum_{i=1}^{N}(-1)^{m(i+1)}\frac{L_{mi-1}}{L_{m(i-1)}L_{mi}L_{m(i+1)}}. \quad (2.43)$$

Then

$$T_9(m,N) = \begin{cases} \frac{F_{m+1}}{2F_mF_{2m}}\left(\frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_m}{L_m}\right) - \frac{F_{mN}}{F_m^2L_mL_{m(N+1)}}, & m \text{ even} \\ \frac{F_{m+1}}{2F_mF_{2m}}\left(\frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_m}{L_m}\right) + \frac{F_{mN}}{F_m^2L_mL_{m(N+1)}}, & m \text{ odd,} \end{cases} \tag{2.44}$$

and

$$T_{10}(m,N) = \begin{cases} \frac{F_{m-1}}{2F_mF_{2m}}\left(\frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_m}{L_m}\right) - \frac{F_{mN}}{F_m^2L_mL_{m(N+1)}}, & m \text{ even} \\ \frac{F_{m-1}}{2F_mF_{2m}}\left(\frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_m}{L_m}\right) + \frac{F_{mN}}{F_m^2L_mL_{m(N+1)}}, & m \text{ odd.} \end{cases} \tag{2.45}$$

Proof. Combine equations (1.3) and (1.8). The proof is essentially the same as done above and is omitted. □

Corollary 2.13.

$$T_9(m,\infty) = \begin{cases} \frac{F_{m+1}}{F_m}\left(\frac{1}{\sqrt{5}F_{2m}} - \frac{1}{2L_m^2}\right) - \frac{1}{\sqrt{5}F_m^2L_m\alpha^m}, & m \text{ even} \\ \frac{F_{m+1}}{F_m}\left(\frac{1}{\sqrt{5}F_{2m}} - \frac{1}{2L_m^2}\right) + \frac{1}{\sqrt{5}F_m^2L_m\alpha^m}, & m \text{ odd} \end{cases} \tag{2.46}$$

and

$$T_{10}(m,\infty) = \begin{cases} \frac{F_{m-1}}{F_m}\left(\frac{1}{\sqrt{5}F_{2m}} - \frac{1}{2L_m^2}\right) - \frac{1}{\sqrt{5}F_m^2L_m\alpha^m}, & m \text{ even} \\ \frac{F_{m-1}}{F_m}\left(\frac{1}{\sqrt{5}F_{2m}} - \frac{1}{2L_m^2}\right) + \frac{1}{\sqrt{5}F_m^2L_m\alpha^m}, & m \text{ odd.} \end{cases} \tag{2.47}$$

Proposition 2.14. *Let m be even such that $m/2$ is even, i.e. $4|m$. Define*

$$T_{11}(m,N) = \sum_{i=1}^{N} \frac{F_{mi-m/2}}{L_{m(i-1)}L_{mi}L_{m(i+1)}}. \tag{2.48}$$

Similarly, for m even with $m/2$ odd define

$$T_{12}(m,N) = \sum_{i=1}^{N} \frac{L_{mi-m/2}}{L_{m(i-1)}L_{mi}L_{m(i+1)}}. \tag{2.49}$$

Then

$$T_{11}(m,N) = \frac{1}{5F_{m/2}}\Big(\frac{1}{2F_{2m}}\Big(\frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_m}{L_m}\Big) - \frac{F_{mN}}{F_mL_mL_{m(N+1)}}\Big), \tag{2.50}$$

and

$$T_{12}(m,N) = \frac{1}{L_{m/2}}\Big(\frac{1}{2F_{2m}}\Big(\frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_m}{L_m}\Big) - \frac{F_{mN}}{F_mL_mL_{m(N+1)}}\Big). \tag{2.51}$$

Proof. The statement is a direct consequence of equation (2.4) of Lemma 2.1. □

Corollary 2.15.

$$T_{11}(m,\infty) = \frac{1}{5F_{m/2}}\Big(\frac{1}{\sqrt{5}F_{2m}} - \frac{1}{2L_m^2} - \frac{1}{\sqrt{5}F_mL_m\alpha^m}\Big), \tag{2.52}$$

and

$$T_{12}(m,\infty) = \frac{1}{L_{m/2}}\Big(\frac{1}{\sqrt{5}F_{2m}} - \frac{1}{2L_m^2} - \frac{1}{\sqrt{5}F_mL_m\alpha^m}\Big). \tag{2.53}$$

Remark 2.16. *For $m = 2$ we have that $T_{12}(2, N) = T_{10}(2, N)$.*

Proposition 2.17. *Let m be even such that $m/2$ is even, i.e. $4|m$. Define*

$$T_{13}(m, N) = \sum_{i=1}^{N} \frac{F_{mi+m/2}}{L_{m(i-1)}L_{m(i+1)}L_{m(i+2)}}. \tag{2.54}$$

Similarly, for m even with $m/2$ odd define

$$T_{14}(m, N) = \sum_{i=1}^{N} \frac{L_{mi+m/2}}{L_{m(i-1)}L_{m(i+1)}L_{m(i+2)}}. \tag{2.55}$$

Then

$$T_{13}(m, N) = \frac{1}{5F_{3m/2}}\Big(\frac{1}{2F_{2m}}\Big(\frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_m}{L_m}\Big) - \frac{F_{mN}}{F_m L_{2m} L_{m(N+2)}}\Big), \tag{2.56}$$

and

$$T_{14}(m, N) = \frac{1}{L_{3m/2}}\Big(\frac{1}{2F_{2m}}\Big(\frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_m}{L_m}\Big) - \frac{F_{mN}}{F_m L_{2m} L_{m(N+2)}}\Big). \tag{2.57}$$

Proof. Set $n = m \geq 1$ in equation (1.3) and combine with equation (1.8). Use equation (2.4) of Lemma 2.1 with $u = mi + m/2$ and $v = 3m/2$. Details are omitted. □

Corollary 2.18.

$$T_{13}(m, \infty) = \frac{1}{5F_{3m/2}}\Big(\frac{1}{\sqrt{5}F_{2m}} - \frac{1}{2L_m^2} - \frac{1}{\sqrt{5}F_m L_{2m}\alpha^{2m}}\Big), \tag{2.58}$$

and

$$T_{14}(m, \infty) = \frac{1}{L_{3m/2}}\Big(\frac{1}{\sqrt{5}F_{2m}} - \frac{1}{2L_m^2} - \frac{1}{\sqrt{5}F_m L_{2m}\alpha^{2m}}\Big). \tag{2.59}$$

Finally, we present the following expressions:

Proposition 2.19. *Let m be even such that $m/2$ is even, i.e. $4|m$. Define*

$$T_{15}(m, N) = \sum_{i=1}^{N} \frac{F_{mi+m/2}}{L_{m(i-1)}L_{mi}L_{m(i+1)}}. \tag{2.60}$$

Similarly, for m even with $m/2$ odd define

$$T_{16}(m, N) = \sum_{i=1}^{N} \frac{L_{mi+m/2}}{L_{m(i-1)}L_{mi}L_{m(i+1)}}. \tag{2.61}$$

Then

$$T_{15}(m, N) = \frac{1}{5F_{m/2}}\Big(\frac{1}{2L_m} + \frac{F_{mN}}{F_m L_m L_{m(N+1)}} - \frac{1}{L_{mN}L_{m(N+1)}} - \frac{1}{2F_{2m}}\Big(\frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_m}{L_m}\Big)\Big), \tag{2.62}$$

and

$$T_{16}(m, N) = \frac{1}{L_{m/2}}\Big(\frac{1}{2L_m} + \frac{F_{mN}}{F_m L_m L_{m(N+1)}} - \frac{1}{L_{mN}L_{m(N+1)}} - \frac{1}{2F_{2m}}\Big(\frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_m}{L_m}\Big)\Big). \tag{2.63}$$

Proof. Set m even and $n = 0$ in (1.3) and combine with (1.8). $\square$

Corollary 2.20.

$$T_{15}(m,\infty) = \frac{1}{5F_{m/2}}\Big(\frac{\sqrt{5}}{10F_m} + \frac{1}{2L_m^2} - \frac{\sqrt{5}}{5F_{2m}}\Big) \tag{2.64}$$

and

$$T_{16}(m,\infty) = \frac{1}{L_{m/2}}\Big(\frac{\sqrt{5}}{10F_m} + \frac{1}{2L_m^2} - \frac{\sqrt{5}}{5F_{2m}}\Big). \tag{2.65}$$

Remark 2.21. *For $m = 2$ we have that $T_{16}(2,N) = T_9(2,N)$.*

We conclude this section with a short list of examples that are contained in the presentation as special cases:

$$T_6(2,1,N) = \sum_{i=1}^{N} \frac{L_{2i}}{L_{2i-1}L_{2i+1}L_{2i+3}} = \frac{11}{60} - \frac{1}{15}\Big(\frac{L_{2N}}{L_{2N+1}} + \frac{L_{2N+2}}{L_{2N+3}}\Big) - \frac{1}{4}\frac{F_{2N}}{L_{2N+3}}, \tag{2.66}$$

$$T_6(2,1,\infty) = \frac{1}{15} + \frac{1}{30\alpha^3}, \tag{2.67}$$

$$T_9(2,N) = \sum_{i=1}^{N} \frac{L_{2i+1}}{L_{2i-2}L_{2i}L_{2i+2}} = \frac{1}{3}\Big(\frac{F_{2N+2}}{L_{2N+2}} + \frac{F_{2N}}{L_{2N}}\Big) - \frac{1}{3}\frac{F_{2N}}{L_{2N+2}} - \frac{1}{9}, \tag{2.68}$$

$$T_9(2,\infty) = \frac{1}{18} + \frac{\sqrt{5}}{30}, \tag{2.69}$$

$$T_9(3,N) = \sum_{i=1}^{N} (-1)^{i+1} \frac{L_{3i+1}}{L_{3i-3}L_{3i}L_{3i+3}} = \frac{3}{32}\Big(\frac{F_{3N+3}}{L_{3N+3}} + \frac{F_{3N}}{L_{3N}}\Big) + \frac{1}{16}\frac{F_{3N}}{L_{3N+3}} - \frac{3}{64}, \tag{2.70}$$

$$T_9(3,\infty) = \frac{1}{64} + \frac{\sqrt{5}}{80}, \tag{2.71}$$

$$T_{11}(4,N) = \sum_{i=1}^{N} \frac{F_{4i-2}}{L_{4i-4}L_{4i}L_{4i+4}} = \frac{1}{210}\Big(\frac{F_{4N+4}}{L_{4N+4}} + \frac{F_{4N}}{L_{4N}}\Big) - \frac{1}{105}\frac{F_{4N}}{L_{4N+4}} - \frac{1}{490}, \tag{2.72}$$

$$T_{11}(4,\infty) = \frac{3}{245} - \frac{\sqrt{5}}{210}, \tag{2.73}$$

$$T_{15}(4,N) = \sum_{i=1}^{N} \frac{F_{4i+2}}{L_{4i-4}L_{4i}L_{4i+4}} = \frac{8}{490} + \frac{1}{5}\Big(\frac{F_{4N}}{21L_{4N+4}} - \frac{1}{L_{4N}L_{4N+4}}\Big) - \frac{1}{210}\Big(\frac{F_{4N+4}}{L_{4N+4}} + \frac{F_{4N}}{L_{4N}}\Big), \tag{2.74}$$

and

$$T_{15}(4,\infty) = \frac{1}{490} + \frac{\sqrt{5}}{210}. \tag{2.75}$$

3 Sums with four factors

It is worth to mention that the formulas derived in this article may be combined once more time to produce closed-form expressions for sums with four factors. We conclude with some explicit examples. In all examples we assume m to be an even integer. Let us define the following series:

$$G_1(m,N)=\sum_{i=1}^{N}\frac{F_{mi+m/2}F_{mi+m}}{L_{m(i-1)}L_{mi}L_{m(i+1)}L_{m(i+2)}}\quad (m/2\ \textit{even}), \tag{3.1}$$

$$G_2(m,N)=\sum_{i=1}^{N}\frac{L_{mi+m/2}F_{mi+m}}{L_{m(i-1)}L_{mi}L_{m(i+1)}L_{m(i+2)}}\quad (m/2\ \textit{odd}), \tag{3.2}$$

$$G_3(m,N)=\sum_{i=1}^{N}\frac{F_{mi+m/2}L_{mi+1}}{L_{m(i-1)}L_{mi}L_{m(i+1)}L_{m(i+2)}}\quad (m/2\ \textit{even}), \tag{3.3}$$

and

$$G_4(m,N)=\sum_{i=1}^{N}\frac{F_{mi+m/2}L_{mi-1}}{L_{m(i-1)}L_{mi}L_{m(i+1)}L_{m(i+2)}}\quad (m/2\ \textit{even}). \tag{3.4}$$

Then the following results hold:

Corollary 3.1.

$$G_1(m,N)=\frac{1}{5F_m}\Big(T_{15}(m,N)-T_{13}(m,N)\Big), \tag{3.5}$$

$$G_2(m,N)=\frac{1}{5F_m}\Big(T_{16}(m,N)-T_{14}(m,N)\Big), \tag{3.6}$$

$$G_3(m,N)=\frac{1}{F_{2m}}\Big(T_{15}(m,N)-F_{2m-1}T_{13}(m,N)\Big), \tag{3.7}$$

and

$$G_4(m,N)=\frac{1}{F_{2m}}\Big(T_{15}(m,N)-F_{2m+1}T_{13}(m,N)\Big). \tag{3.8}$$

These results are also valid for $N\to\infty$.

Proof. Combine the series from the previous section and use Lemma 2.1. □

References

[1] Almkvist, G. (1986) A Solution to a Tantalizing Problem, *The Fibonacci Quarterly*, 24 (4), 316–322.

[2] André-Jeannin, R. (1990) Lambert Series and the Summation of Reciprocals of Certain Fibonacci-Lucas-Type Sequences, *The Fibonacci Quarterly*, Vol. 28 (3), 223–226.

[3] Backstrom, R.P (1981)On Reciprocal Series Related to Fibonacci Numbers with Subscripts in Arithmetic Progression, *The Fibonacci Quarterly*, 19, 14–21.

[4] Borwein, J.M. & Borwein, P.B. (1987) *Pi and the AGM,* John Wiley & Sons, New York.

[5] Brousseau, B.A. (1969) Summation of Infinite Fibonacci Series, *The Fibonacci Quarterly*, 7 (2), 143–168.

[6] Carlitz, L. (1971) Reduction Formulas for Fibonacci Summations, *The Fibonacci Quarterly*, 9(5), 449–466, 510–511.

[7] Frontczak, R. (2016) A Note on a Family of Alternating Fibonacci Sums, *Notes on Number Theory and Discrete Mathematics* 22(2), 64–71.

[8] Frontczak, R. (2017) Further Results on Arctangent Sums with Applications to Generalized Fibonacci Numbers, *Notes on Number Theory and Discrete Mathematics*, 23(1), 39–53.

[9] Frontczak, R. (2017) Summation of Some Finite Fibonacci Lucas Series, *unpublished manuscript*.

[10] Good, I. J., (1974) A reciprocal series of Fibonacci numbers, *The Fibonacci Quarterly*, 12, 346.

[11] Horadam, A. F. (1988) Elliptic Functions and Lambert Series in the Summation of Reciprocals in Certain Recurrence-Generated Sequences, *The Fibonacci Quarterly*, 26(2), 98–114.

[12] Hu, H., Sun, Z.-W., & Liu, J.-X. (2001) Reciprocal sums of second-order recurrent sequences, *The Fibonacci Quarterly*, 39, 214–220.

[13] Melham, R. S. (1999) Lambert Series and Elliptic Functions and Certain Reciprocal Sums, *The Fibonacci Quarterly*, 37, 208–212.

[14] Melham, R. S. (2000) Summation of reciprocals which involve products of terms from generalized Fibonacci sequences, *The Fibonacci Quarterly*, 38, 294–298.

[15] Melham, R. S. (2001) Summation of reciprocals which involve products of terms from generalized Fibonacci sequences - Part II, *The Fibonacci Quarterly*, 39, 264–267.

[16] Melham, R. S. (2002) Reduction Formulas for the Summation of Reciprocals in Certain Second-Order Recurring Sequences, *The Fibonacci Quarterly*, 40(1), 71–75.

[17] Melham, R. S. (2003) On Some Reciprocal Sums of Brousseau: An Alternative Approach to that of Carlitz, *The Fibonacci Quarterly*, 41(1), 59–62.

[18] Melham, R. S. (2012) On finite sums of Good and Shar that involve reciprocals of Fibonacci numbers, *Integers,* Vol. 12 (A61).

[19] Melham, R. S.(2013) Finite Sums That Involve Reciprocals of Products of Generalized Fibonacci Numbers, *Integers,* Vol. 13 (A40).

[20] Melham R. S. (2014) More on Finite Sums that Involve Reciprocals of Products of Generalized Fibonacci Numbers, *Integers,* Vol. 14 (A4).

[21] Melham, R. S. (2015) On Certain Families of Finite Reciprocal Sums that Involve Generalized Fibonacci Numbers, *The Fibonacci Quarterly*, 53(4), 323–334.

[22] Melham, R. S., & Shannon, A. G. (1995) On Reciprocal Sums of Chebyshev Related Sequences, *The Fibonacci Quarterly*, 33(3), 194–202.

[23] Rabinowitz, S. (1999) Algorithmic Summation of Reciprocals of Products of Fibonacci Numbers, *The Fibonacci Quarterly*, 37, 122–127.