Notes on Number Theory and Discrete Mathematics
Print ISSN 1310-5132, Online ISSN 2367-8275
Vol. 23, 2017, No. 2, 91–103

On the Pell p-circulant sequences

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Received: 11 January 2016 **Accepted:** 31 January 2017

Abstract: In this paper, we define the generalized Pell p-circulant sequence and the Pell p-circulant sequence by using the circulant matrices which are obtained from the characteristic polynomial of the generalized Pell (p,i)-sequence and then, we obtain miscellaneous properties of these sequences. Also, we consider the cyclic groups which are generated by the generating matrices and the auxiliary equations of the defined recurrence sequences and then, we study the orders of these groups. Furthermore, we extend the Pell p-circulant sequence to groups. Finally, we obtain the lengths of the periods of Pell p-circulant sequences in the semidihedral group SD_{2^m} for $m \geq 4$ as applications of the results obtained.

Keywords: Circulant Matrix, Sequence, Group, Period.
AMS Classification: 11B50, 20F05, 15A36, 20D60.

1 Introduction

Kilic [17] defined the generalized Pell (p,i)-numbers as follows:

for any given $p\,(p = 1,2,3,\ldots)$ $n > p+1$ and $0 \leq i \leq p$

$$P_p^{(i)}(n) = 2P_p^{(i)}(n-1) + P_p^{(i)}(n-p-1) \tag{1.1}$$

with initial conditions $P_p^{(i)}(1) = \cdots = P_p^{(i)}(i) = 0$ and

$$P_p^{(i)}(i+1) = P_p^{(i)}(i+2) = \cdots = P_p^{(i)}(p+1) = 1.$$

It is clear that the characteristic polynomial of the generalized Pell (p,i)-sequence is

$$f(x) = x^{p+1} - 2x^p - 1.$$

Davis [4] defined the circulant matrix $C_n = \left[c_{ij}\right]_{n\times n}$ associated with the numbers $c_0, c_1, \cdots, c_{n-1}$ as follows:

$$C_n = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & \cdots & c_3 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-3} & \cdots & c_0 & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}.$$

The $(n-1)$th degree polynomial $P(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$ is called the associated polynomial of the circulant matrix C_n [cf.2,15,20,22,24].

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding *k* terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants.

Kalman [16] derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

He also showed that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

These have been used by Gray in the development of the related theory Toeplitz matrices [13], and by Shannon and Bernstein in an extension of a generalization of continued fractions to arbitrary order recursive sequences [23].

In Section 2, we define the generalized Pell p-circulant sequence and the Pell p-circulant sequence such that these sequences are obtained from the circulant matrix C_{P+2} which is defined by using the characteristic polynomial of the generalized Pell (p,i)-sequence. Then we obtain their miscellaneous properties.

In [5,6,7,8,9,19], the authors obtained the cyclic groups and the semigroups via some special matrices. In Section 3, we consider the multiplicative orders of the circulant matrix C_{P+2} and the Pell p-circulant matrix M_{P+2} working to modulo *m* and then, we obtain the cyclic groups which are generated by reducing these matrices modulo *m*. Also in this section, we study the defined recurrence sequences modulo m. Then we derive the relationships between the orders of the obtained cyclic groups and the periods of the defined sequences according to modulo *m*.

The study of recurrence sequences in groups began with the earlier work of Wall [26] where the ordinary Fibonacci sequences in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by several authors; see for example, [1,3,5,7,9,10,11,18,19,21,25]. In Section 4, we define the Pell p-circulant sequence by means of the elements of the groups which have two or more generators, and then we examine this sequence in finite groups. Furthermore, we examine the behaviours of the lengths of the periods of the Pell p-circulant sequences in the semidihedral group SD_{2^m} for $m \geq 4$.

2 The Generalized Pell p-Circulant and The Pell p-Circulant Sequences

We can write the following circulant matrix for the polynomial $f(x)$ is as follows:

$$C_{P+2} = \left[C_{ij}\right]_{(P+2)(P+2)} = \begin{cases} -1 & \text{if } (i=j), \\ 1 & \text{if } (i=j+1) \text{ and } (i=p+2,\, j=1), \\ -2 & \text{if } (i=j+2), (i=p+1,\, j=1) \text{ and } (i=p+2,\, j=2), \\ 0 & \text{otherwise.} \end{cases}$$

For example, the matrix C_4 is as follows:

$$C_4 = \begin{bmatrix} -1 & 1 & -2 & 0 \\ 0 & -1 & 1 & -2 \\ -2 & 0 & -1 & 1 \\ 1 & -2 & 0 & -1 \end{bmatrix}.$$

Define the generalized Pell p-circulant sequence by using the matrices C_{p+2} as shown:

$$x_n = \begin{cases} -2x_{n-p} + x_{n-p-1} - x_{n-p-2}, & n \equiv 1 \bmod (p+2), \\ \vdots & \\ -2x_{n-p} + x_{n-p-1} - x_{n-p-2}, & n \equiv p \bmod (p+2), \quad \text{for } n > p+2, \\ x_{n-p-1} - x_{n-p-2} - 2x_{n-2p-2}, & n \equiv p+1 \bmod (p+2), \\ -x_{n-p-2} - 2x_{n-2p-2} + x_{n-2p-3} & n \equiv 0 \bmod (p+2) \end{cases} \tag{2.1}$$

where $x_1 = x_2 = \cdots = x_{p+1} = 0$ and $x_{p+2} = 1$.

For $n \geq 0$, by an inductive argument, we may write

$$\left(C_{p+2}\right)^n = \begin{bmatrix} x_{n(p+2)+p+2} & x_{n(p+2)+p+1} & x_{n(p+2)+p} & \cdots & x_{n(p+2)+2} & x_{n(p+2)+1} \\ x_{n(p+2)+1} & x_{n(p+2)+p+2} & x_{n(p+2)+p+1} & \cdots & x_{n(p+2)+3} & x_{n(p+2)+2} \\ x_{n(p+2)+2} & x_{n(p+2)+1} & x_{n(p+2)+p+2} & \cdots & & x_{n(p+2)+3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ x_{n(p+2)+p} & x_{n(p+2)+p-1} & x_{n(p+2)+p-2} & \cdots & x_{n(p+2)+p+2} & x_{n(p+2)+p+1} \\ x_{n(p+2)+p+1} & x_{n(p+2)+p} & x_{n(p+2)+p-1} & \cdots & x_{n(p+2)+1} & x_{n(p+2)+p+2} \end{bmatrix}. \tag{2.2}$$

It is easy to see that $\left(C_{p+2}\right)^n$ is a circulant matrix of order $p+2$.

We next define the Pell p-circulant sequence as

$$a_{n+p+2} = a_{n+p+1} - 2a_{n+p} - a_n \qquad \text{for } n > 0 \tag{2.3}$$

where $a_1 = a_2 = \cdots = a_{p+1} = 0, a_{p+2} = 1$ and $p \geq 2$.

We then obtain that the generating function of the Pell p-circulant sequence $\{a_n\}$ is as follows:

$$g(x) = \frac{x^{p+1}}{x^{p+2} + 2x^2 - x + 1}.$$

By (2.3), we can write the following companion matrix:

$$M_{P+2} = \left[m_{ij}\right]_{p+2 \times p+2} = \begin{bmatrix} 1 & -2 & 0 & \dots & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

The matrix M_{P+2} is said to be the Pell p-circulant matrix. It is clear that

$$\begin{bmatrix} a_{n+p+2} \\ a_{n+p+1} \\ \vdots \\ a_{n+1} \end{bmatrix} = M_{p+2} \begin{bmatrix} a_{n+p+1} \\ a_{n+p} \\ \vdots \\ a_n \end{bmatrix}$$

For $n \geq p+1$, by an inductive argument, we may write

$$
\left(M_{p+2}\right)^n = \begin{bmatrix} a_{n+p+2} & -2a_{n+p+1} - a_{n+1} & -a_{n+2} & -a_{n+3} & \cdots & -a_{n+p+1} \\ a_{n+p+1} & -2a_{n+p} - a_n & -a_{n+1} & -a_{n+2} & \cdots & -a_{n+p} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{n+2} & -2a_{n+1} - a_{n-p+1} & -a_{n-p+2} & -a_{n-p+3} & \cdots & -a_{n+1} \\ a_{n+1} & -2a_n - a_{n-p} & -a_{n-p+1} & -a_{n-p+2} & \cdots & -a_n \end{bmatrix}. \quad (2.4)
$$

Note that $\det M_{p+2} = (-1)^p$. It is well-known that the Simson identity for a recursive sequence can be obtained from the determinant of its generating matrix.

Lemma 2.1. The characteristic equation of the Pell p-circulant sequence $x^{p+2} - x^{p+1} + 2x^p + 1 = 0$ does not have multiple roots.

Proof. Let

$$h(z) = z^{p+2} - z^{p+1} + 2z^p + 1$$

and let μ be a multiple root of $h(z)$. Then $h(\mu) = 0$ and $h^{'}(\mu) = 0$. We first note that 0 is not a root of $h(z)$. So, we obtain

$$h^{'}(\mu) = (p+2)\mu^{p+1} - (p+1)\mu^p + 2p\mu^{p-1} = \mu^{p-1}\left((p+2)\mu^2 - (p+1)\mu + 2p\right) = 0.$$

Thus

$$\mu_{1,2} = \frac{(p+1) \pm \sqrt{-7p^2 - 14p + 1}}{2p+4}$$

and hence,

$$h(\mu_1) = \left(\frac{(p+1) + \sqrt{-7p^2 - 14p + 1}}{2p+4}\right)^p \left(\left(\frac{(p+1) + \sqrt{-7p^2 - 14p + 1}}{2p+4}\right)^2 - \left(\frac{(p+1) + \sqrt{-7p^2 - 14p + 1}}{2p+4}\right) + 2\right) = -1$$

and

$$h(\mu_2) = \left(\frac{(p+1) - \sqrt{-7p^2 - 14p + 1}}{2p+4}\right)^p \left(\left(\frac{(p+1) - \sqrt{-7p^2 - 14p + 1}}{2p+4}\right)^2 - \left(\frac{(p+1) - \sqrt{-7p^2 - 14p + 1}}{2p+4}\right) + 2\right) = -1$$

Since $p \geq 2$, by induction on p, it is see that $h(\mu_1) \neq -1$ and $h(\mu_2) \neq -1$, which are contradictions. Thus, the equation $h(\mu) = 0$ does not have multiple roots. □

Let $h(\varepsilon)$ be the characteristic polynomial of the Pell p-circulant matrix M_{p+2}. If $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{p+2}$ are eingenvalues of the matrix M_{p+2}, then by Lemma 2.1, it is known that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{p+2}$ are distinct. Let V be a $(p+2)\times(p+2)$ Vandermonde matrix:

$$V = \begin{bmatrix} \varepsilon_1^{p+1} & \varepsilon_2^{p+1} & \dots & \varepsilon_{p+2}^{p+1} \\ \varepsilon_1^{p} & \varepsilon_2^{p} & \dots & \varepsilon_{p+2}^{p} \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_{p+2} \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Suppose that

$$W^i = \begin{bmatrix} \varepsilon_1^{n+p+2-i} \\ \varepsilon_2^{n+p+2-i} \\ \vdots \\ \varepsilon_{p+2}^{n+p+2-i} \end{bmatrix}$$

and $V^{(i,j)}$ is a $(p+2)\times(p+2)$ matrix obtained from V by replacing the jth column of V by W^i. We can now establish the Binet formula for the Pell p-circulant sequence with the following Theorem.

Theorem 2.1. Let a_n be the nth term of the Pell p-circulant sequence. Then

$$m_{ij}^{(n)} = \frac{\det\left(V^{(i,j)}\right)}{\det(V)}$$

where

$$\left(M_{p+2}\right)^n = \left[m_{ij}^{(n)}\right].$$

Proof. Since the eigenvalues of the matrix M_{p+2} are distinct, the matrix M_{p+2} is diagonalizable. Let $D = \operatorname{diag}\left(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{p+2}\right)$, then it is easy to see that $M_{p+2}V = VD$. Since the matrix V is invertible, $(V)^{-1} M_{p+2}V = D$. Hence, the matrix M_{p+2} is similar to D. So we get $\left(M_{p+2}\right)^n V = VD^n$ for $n \geq 1$. Then we write the following linear system of equations for:

$$\begin{aligned} &m_{i1}^{(n)}\varepsilon_1^{p+1} + m_{i2}^{(n)}\varepsilon_1^{p} + \cdots + m_{ip+2}^{(n)} = \varepsilon_1^{n+p+2-i} \\ &m_{i1}^{(n)}\varepsilon_2^{p+1} + m_{i2}^{(n)}\varepsilon_2^{p} + \cdots + m_{ip+2}^{(n)} = \varepsilon_2^{n+p+2-i} \\ &\vdots \\ &m_{i1}^{(n)}\varepsilon_{p+2}^{p+1} + m_{i2}^{(n)}\varepsilon_{p+2}^{p} + \cdots + m_{ip+2}^{(n)} = \varepsilon_{p+2}^{n+p+2-i}. \end{aligned}$$

So, we obtain

$$m_{ij}^{(n)} = \frac{\det\left(V^{(i,j)}\right)}{\det(V)} \qquad \text{for } i, j = 1, 2, \dots, p+2.$$

□

3 The Cyclic Groups via the matrices C_{P+2} and M_{P+2}

For given a matrix $A=\left[a_{ij}\right]$ with m_{ij}'s integers, $A\left(\bmod m\right)$ means that each element of A is reduced modulo m, that is, $A\left(\bmod\ m\right)=\left(a_{ij}\left(\bmod\ m\right)\right)$. Let us consider the set $\left\langle A\right\rangle_m=\left\{A^i\left(\bmod m\right)\middle| i\ge 0\right\}$. If $\gcd\left(m,\det A\right)=1$, then the set $\left\langle A\right\rangle_m$ is a cyclic group. Let the notation $\left|\left\langle A\right\rangle_m\right|$ denotes the order of the set $\left\langle A\right\rangle_m$. Since $\det M_{p+2}=\left(-1\right)^p$, it is clear that the set $\left\langle M_{p+2}\right\rangle_m$ is acyclic group for every positive integer m. Similarly, the set $\left\langle C_{p+2}\right\rangle_m$ is a cyclic group if $\gcd\left(m,C_{p+2}\right)=1$. We next consider the cyclic groups generated by these matrices C_{P+2} and M_{P+2}.

Theorem 3.1. Let β be a prime and let $\left\langle G\right\rangle_{\beta^n}$ be any of the cyclic groups $\left\langle C_{p+2}\right\rangle_{\beta^n}$ and $\left\langle M_{p+2}\right\rangle_{\beta^n}$ such that $n\in N$. If u is the largest positive integer such that $\left|\left\langle G\right\rangle_\beta\right|=\left|\left\langle G\right\rangle_{\beta^u}\right|$, then $\left|\left\langle G\right\rangle_{\beta^v}\right|=\beta^{v-u}\cdot\left|\left\langle G\right\rangle_\beta\right|$ for every $v\ge u$. In particular, if $\left|\left\langle G\right\rangle_\beta\right|\ne\left|\left\langle G\right\rangle_{\beta^2}\right|$, then $\left|\left\langle G\right\rangle_\beta\right|=\beta^{v-1}\cdot\left|\left\langle G\right\rangle_\beta\right|$ for every $v\ge 2$.

Proof. Let us consider the cyclic group $\left\langle M_{p+2}\right\rangle_{\beta^n}$ and let $\left|\left\langle M_{P+2}\right\rangle_{\beta^n}\right|$ be denoted by $h\left(\beta^n\right)$. If $\left(M_{P+2}\right)^{h\left(\beta^{a+1}\right)}\equiv I\left(\bmod \beta^{a+1}\right)$, then $\left(M_{P+2}\right)^{h\left(\beta^{a+1}\right)}\equiv I\left(\bmod \beta^{a}\right)$ where a is a positive integer and I is a $\left(p+2\right)\times\left(p+2\right)$ identity matrix. Thus we find that $h\left(\beta^a\right)$ divides $h\left(\beta^{a+1}\right)$. Also, writing $\left(M_{p+2}\right)^{h\left(\beta^a\right)}=I+\left(m_{ij}{}^{(a)}\cdot\beta^a\right)$ we get from the binomial expansion that

$$\left(M_{P+2}\right)^{h\left(\beta^a\right)\cdot\beta}=\left(I+\left(m_{ij}{}^{(a)}\cdot\beta^a\right)\right)^\beta=\sum_{i=0}^{\beta}\binom{\beta}{i}\left(m_{ij}{}^{(a)}\cdot\beta^a\right)^i\equiv I\left(\bmod \beta^{a+1}\right),$$

which yields that $h\left(\beta^{a+1}\right)$ divides $h\left(\beta^a\right)\cdot\beta$. Thus, $h\left(\beta^{a+1}\right)=h\left(\beta^a\right)$ or $h\left(\beta^{a+1}\right)=h\left(\beta^a\right)\cdot\beta$. It is clear then that $h\left(\beta^{a+1}\right)=h\left(\beta^a\right)\cdot\beta$ holds if and only if there exists an integer $m_{ij}^{(a)}$ which is not divisible by β. Since u is the largest positive integer such that $h\left(\beta\right)=h\left(\beta^u\right)$, we have $h\left(\beta^u\right)\ne h\left(\beta^{u+1}\right)$. Then, there exists an integer $m_{ij}{}^{(u+1)}$ which is not divisible by β. So we get that $h\left(\beta^{u+1}\right)\ne h\left(\beta^{u+2}\right)$. To complete the proof we may use an inductive method on u.
There is a similar proof for the cyclic group $\left\langle C_{p+2}\right\rangle_{\lambda^m}$. □

Theorem 3.2. Let $\langle G\rangle_m$ be any of the cyclic groups $\langle C_{P+2}\rangle_m$ and $\langle M_{p+2}\rangle_m$ and let $m=\prod_{i=1}^{t}\beta_i^{e_i},(t\geq 1)$ where β_i's are distinct primes. Then

$$\left|\langle G\rangle_m\right| = \mathrm{lcm}\left[\langle G\rangle_{\beta_1^{e_1}},\langle G\rangle_{\beta_2^{e_2}},\ldots,\langle G\rangle_{\beta_t^{e_t}}\right].$$

Proof. Let us consider the cyclic group $\langle C_{P+2}\rangle_m$. Suppose that $\left|\langle C\rangle_{\beta_i^{e_i}}\right|=\alpha_i$ for $1\leq i\leq t$ and let $\left|\langle C_{p+2}\rangle_m\right|=\alpha$. Then by (2.2), we obtain

$$a_{\alpha_i(p+2)+j}\equiv 0 \bmod \beta_i^{e_i} \quad \text{for } 1\leq j\leq p+1,$$
$$a_{\alpha_i(p+2)+p+2}\equiv 1 \bmod \beta_i^{e_i}$$

and

$$a_{\alpha_i(p+2)+j}\equiv 0 \bmod m \quad \text{for } 1\leq j\leq p+1,$$
$$a_{\alpha_i(p+2)+p+2}\equiv 1 \bmod m.$$

This implies that $a_{\alpha(p+2)+p+2}=k\cdot a_{\alpha_i(p+2)+j}$ for $1\leq j\leq p+2$ and $k\in N$ that is, $(C_{P+2})^{\alpha}$ is of the form $k\cdot(C_{P+2})^{\alpha_i}$ for all values of i. Thus it is verified that

$$\left|\langle C_{p+2}\rangle_m\right| = \mathrm{lcm}\left[\langle C_{p+2}\rangle_{\beta_1^{e_1}},\langle C_{p+2}\rangle_{\beta_2^{e_2}},\ldots,\langle C_{p+2}\rangle_{\beta_t^{e_t}}\right].$$

There is similar proof for the set $\langle M_{p+2}\rangle_m$. □

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period *k* if the first *k* elements in the sequence form a repeating subsequence.

Reducing the generalized Pell p-circulant sequence and the Pell p-circulant sequence of the by a modulus m, we can get the repeating sequences, respectively denoted by

$$\{x_n(m)\}=\{x_1(m),x_2(m),x_3(m),\ldots,x_j(m),\ldots\}$$

and

$$\{a_n(m)\}=\{a_1(m),a_2(m),a_3(m),\ldots,a_j(m),\ldots\},$$

where $x_j(m)=x_j(\bmod m)$, $a_j(m)=a_j(\bmod m)$. They have the same recurrence relation as in the definitions of the generalized Pell p-circulant sequence and the Pell p-circulant sequence, respectively.

Theorem 3.3. The sequence $\{a_n(m)\}$ is simply periodic for every positive integer m. Similarly, the sequence $\{x_n(m)\}$ is a simply periodic sequence if $\gcd(\det C_{p+2},m)=1$.

Proof. Let us consider the Pell p-circulant sequence $\{a_n(m)\}$. Let $Q=\{(q_1,q_2,\ldots,q_{p+2}) \mid 0\le q_i\le m-1\}$, then $|Q|=m^{P+2}$. Since there are m^{P+2} distinct $p+2$-tuples of elements of $\mathbb{Z}_m$, at least one of the $p+2$-tuples appears twice in the sequence $\{a_n(m)\}$. Thus, the subsequence following this $p+2$-tuple repeats; that is the sequence is periodic. So if $a_{i+1}(m)\equiv a_{j+1}(m)$, $a_{i+2}(m)\equiv a_{j+2}(m),\ldots,$ $a_{i+p+2}(m)\equiv a_{j+p+2}(m)$ and $i>j$, then $i\equiv j \bmod p+2$. From the definition, we can easily derive that

$$a_i(m)\equiv a_j(m),\ a_{i-1}(m)\equiv a_{j-1}(m),\ \ldots,\qquad a_{i-(j-1)}(m)\equiv a_{j-(j-1)}(m)=a_1(m).$$

Thus we get that the sequence is a simply periodic.
There is a similar proof for the sequence $\{x_n(m)\}$. □

We denote the lengths of the periods of the sequences $\{x_n(m)\}$ and $\{a_n(m)\}$ by $l_x^p(m)$, $l_a^p(m)$, respectively. Then, we have the following useful results from (2.2) and (2.4), respectively.

Corollary 3.1. Let β be a prime. Then
i. If $\gcd(\det C_{p+2},\beta)=1$, then $l_x^p(\beta)=(p+2).\left|\langle C_{P+2}\rangle_\beta\right|$.
ii. $l_a^p(\beta)=\left|\langle M_{p+2}\rangle_\beta\right|$. □

Let β be a prime and let

$$A(\beta^k)=\{x^n(\bmod \beta^k): n\in\mathbb{Z},\ x^{p+2}=x^{p+1}-2x^p-1\}$$

such that $k\in N$. Then, it is clear that the set $A(\beta^k)$ is a cyclic group.
Now we can give a relationship between the characteristic equation of the Pell p-circulant sequence and the period $l_a^p(m)$ by the following Corollary.

Corollary 3.2. Let β be a prime and let $k\in N$. Then, the cyclic group $A(\beta^k)$ is isomorphic to the cyclic group $\langle M_{p+2}\rangle_{\beta^k}$. □

4 The Pell p-Circulant Sequence in Groups

Let G be a finite *j*-generator group and let X be the subset of $\underbrace{G\times G\times G\times\cdots\times G}_{j}$ such that $(x_1,x_2,\ldots,x_j)\in X$ if and only if G is generated by $x_1,x_2,\ldots,x_j$. We call $(x_1,x_2,\ldots,x_j)$ a generating *j*-tuple for G.

Definition 4.1. Let $G=\langle X\rangle$ be a finitely generated group such that $X=\{x_1,x_2,\ldots,x_j\}$. Then we define the Pell p-circulant sequence in the group G as follows:

$$a_{n+j}=\begin{cases}(a_{n+j-2})^{-2}(a_{n+j-1}) & \text{if } n+j\le p+2,\\ (a_{n+j-p-2})^{-1}(a_{n+j-2})^{-2}(a_{n+j-1}) & \text{if } n+j>p+2\end{cases}$$

for $n\ge 1$, with initial conditions

$$a_k=x_k \quad \text{for} \quad 1\le k\le j.$$

For a *j*-tuple $(x_1,x_2,\ldots,x_j)\in X$, the Pell p-circulant sequence in a group G is denoted by $PC^p_{(x_1,x_2,\ldots,x_j)}(G)$.

Theorem 4.1. A Pell p-circulant sequence in a finite group is simply periodic.

Proof. Suppose that n is the order of G. Since there n^{p+2} distinct $p+2$-tuples of elements of G, at least one of the $p+2$-tuples appears twice in the sequence $PC^p_{(x_1,x_2,\ldots,x_j)}(G)$. Thus, consider the subsequence following this $p+2$-tuple. Because of the repeating, the sequence is periodic. Since the sequence $PC^p_{(x_1,x_2,\ldots,x_j)}(G)$ is periodic, there exist natural numbers u and v, with $u\ge v$, such that

$$a_{u+1}=a_{v+1},\ a_{u+2}=a_{v+2},\ldots,a_{u+p+2}=a_{v+p+2}.$$

By the definition of the sequence $PC^p_{(x_1,x_2,\ldots,x_j)}(G)$, we know that

$$a_{n+j-1}=\begin{cases}(a_{n+j-2})^{2}(a_{n+j}) & \text{if } n+j\le p+2,\\ (a_{n+j-2})^{2}(a_{n+j-p-2})(a_{n+j}) & \text{if } n+j>p+2\end{cases}.$$

Therefore, we obtian $a_u=a_v$, and hence,

$$a_{u-v+1}=a_1,a_{u-v+2}=a_2,\ldots,a_{u-v+p+2}=a_{p+2},$$

which implies that the sequence is a simply periodic sequence. □

Let $LPC^p_{(x_1,x_2,\ldots,x_j)}(G)$ denote the length of the period of the sequence $PC^p_{(x_1,x_2,\ldots,x_j)}(G)$. From the definition of the sequence $PC^p_{(x_1,x_2,\ldots,x_j)}(G)$ it is clear that the period of this sequence in a finite group depends on the chosen generating set and the order in which the assignments of $x_1,x_2,\ldots,x_j$ are made.

We shall now address the lengths of the periods the Pell p-circulant sequences in the semidihedral group SD_{2^m}. A group SD_{2^m} is semidihedral group of order 2^m if

$$SD_{2^m} = \left\langle a,b \middle| a^{2^{m-1}} = b^2 = e, b^{-1}ab = a^{-1+2^{m-2}} \right\rangle$$

for every $m \geq 4$. Note that the orders a and b are 2^{m-1} and 2, respectively [cf. 12,14].

Theorem 4.2. The lengths of the periods the Pell p-circulant sequences in the semidihedral group SD_{2^m} are as follows:

i. $LPC^2\left(SD_{2^m}; a, b\right) = 2^{m-3} \cdot l_a^2(2)$ for $p = 2$,

ii. $LPC^p\left(SD_{2^m}; a, b\right) = 2^{m-2} \cdot l_a^p(2)$ for $p \geq 3$.

Proof. **i.** We prove this by direct calculation. Note that $|a| = 2^{m-1}$, $|b| = 2$ and $l_a^2(2) = 15$. Then, the group SD_{2^m} is defined by the presentation

$$\left\langle a,b \middle| a^{2^{m-1}} = b^2 = e, b^{-1}ab = a^{-1+2^{m-2}} \right\rangle.$$

It is clear that $ab = ba^{-1+2^{m-2}}$. The sequence $PC^2\left(SD_{2^m}; a, b\right)$ is

$$a, b, a^{-2}b, a^{-2}b, a^{-3}b, a^{\left(2^{m-1}-3\right)\left(2^{m-2}-1\right)}, ba^5, a^{\left(2^{m-1}-7\right)}, ba^{3+\left(2^{m-1}-7\right)+\left(2^{m-2}\right)},$$
$$a^{-1}b, a^{\left(2^{m-2}+12\right)}, a^3, ba^{\left(2^{m-2}+7\right)}, a^{\left(2^{m-1}-4\right)}, e.$$

Using the above, the sequence becomes:

$$x_1 = a, x_2 = b, x_3 = a^{-2}b, x_4 = a^{-2}b, \ldots,$$
$$x_{16} = a^5, x_{17} = a^4b, x_{18} = a^{-2}b, x_{19} = a^{-2}b, , \ldots,$$
$$x_{15i+1} = a^{1+4i}, x_{15i+2} = a^{4i}b, x_{15i+3} = a^{-2}b, x_{15i+4} = a^{-2}b, \ldots.$$

So we need the smallest integer i such that $4i = 2^{m-1} \cdot k_1$ for $k_1 \in N$. It is easy to see that the length of the period of the sequence is $2^{m-3} \cdot 15$. □

ii. For $p \geq 3$, the sequence $PC^p\left(SD_{2^m}; a, b\right)$ has the following form:

From above, the sequence becomes:

$$x_1 = a, x_2 = b, x_3 = a^{-2}b, x_4 = a^{-2}b, \ldots, x_{p+2} = a^{-2}b, \ldots,$$
$$x_{2 \cdot l_a^p(2) \cdot j+1} = a^{1+\left(2^{m-1}-4\right)j}, x_{2 \cdot l_a^p(2) \cdot j+2} = b, x_{2 \cdot l_a^p(2) \cdot j+3} = a^{-2+\left(2^{m-1}-4\right)j \cdot \lambda_1}b,$$
$$x_{2 \cdot l_a^p(2) \cdot j+4} = a^{-2+\left(2^{m-1}-4\right)j \cdot \lambda_2}b, \ldots, x_{2 \cdot l_a^p(2) \cdot j+p+2} = a^{-2+\left(2^{m-1}-4\right)j \cdot \lambda_p}b, \ldots,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p \in N$. So we need the smallest integer j such that $\left(2^{m-1}-4\right) j = 2^{m-1} k_2$. Thus, we get that $LPC^p\left(SD_{2^m}; a, b\right) = 2^{m-2} \cdot l_a^p(2)$. □

This completes the outline of the algebraic properties of the generalized Pell p-circulant sequences which we sought to develop.

Acknowledgment
This Project was supported by the Commission for the Scientific Research Projects of Kafkas University. The Project number. 2014-FEF-34

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