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A conjecture on degrees of algebraic equations

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Abstract: It is proven that the validity of a conjecture on the degrees of an algebraic equation consisting of three polynomials is determined by the derivatives. The result is extended to positive polynomials satisfying a generalized Fermat equation, after setting the exponents $X,\ Y$ and Z equal to 1, and specialization to prime factors of the product of integer values of the polynomials yields the inequality equivalent to the abc conjecture.
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1 Introduction

The roots of an algebraic equation may be related to those of the derivatives. The averages are equal, and there are equations relating directly the first three moments of the distributions of zeros [1, 2]. The convex hulls of the zeros of the derivatives of polynomials in the complex plane are subsets of the convex hull of the initial polynomial by the Gauss–Lucas theorem [3, 4].

The derivatives will be used here to establish a theorem on the maximum degree of an algebraic equation consisting of three polynomials. Each of these polynomials may have a different degree. Since the derivative reduces the degrees of the polynomials by one, a result for a sequence of equations may be derived. A minimization of the overlap of the sets of zeros of the derivatives occurs when the combined roots of m^{th} derivative of the three polynomials form a maximal set for some m. It is proven in Theorem 1 that, if the maximum degree of three polynomials satisfying $a(x) + b(x) = c(x)$ is larger than one third of the sum of the numbers of roots, the difference will decrease to zero if there is a non-zero overlap between the roots of the derivatives until the coincidences no longer occur if $\deg\ a(x) = \deg\ b(x) = \deg\ c(x)$.

The coincident roots of $a(x)$ and $b(x)$ also would be zeros of $c(x)$. Therefore the convex hull of this subset of roots of $c(x)$ would be identified with the hull of the same subset for $a(x)$. Similarly, the coincident roots of $a^{(m)}(x)$ and $b^{(m)}(x)$ for any m also would be zeros of $c^{(m)}(x)$. Again, the convex hull with vertices at this subset of zeros would be equivalent for $a^{(m)}(x)$, $b^{(m)}(x)$ and $c^{(m)}(x)$. Beyond this subset of the convex hulls of zeros of $c(x)$ and $c^{(m)}(x)$, it is conjectured that there would be relation between the remaining portions defined by the two sets of distinct roots.

This polynomial equation represents the equivalent condition on the coprime integers in the abc conjecture. Since the existence of polynomials satisfying the analogous conjecture has been established [5], the specialization to specific values satisfying $a + b = c$ and the validity of the resulting inequality for c will be sufficient to demonstrate the abc conjecture.

2 Derivatives of an algebraic equation consisting of three polynomials

The maximum number of combined roots of $a(x)$, $b(x)$ and $c(x)$, where

$$a(x) + b(x) = c(x), \tag{2.1}$$

would be three times the highest degree of these polynomials [6, 7], with coincident zeros being counted only once. If the derivative of the relation is considered,

$$a'(x) + b'(x) = c'(x), \tag{2.2}$$

the result holds with the highest degree of $a'(x)$, $b'(x)$ and $c'(x)$ being less by one. However, the roots of $a'(x)$, $b'(x)$ and $c'(x)$ would comprise a set with a number of elements that has decreased by a maximum of three. Consistency of the theorem requires either the overlap between the sets of zeros of $a'(x)$, $b'(x)$ and $c'(x)$ is decreased as a result of the derivative or the maximum value of the set of combined roots had not been not achieved for $a(x)$, $b(x)$ and $c(x)$.

Theorem 1. *If the maximal degree of the three polynomials in the equation $a(x)+b(x) = c(x)$ is greater than one third of the sums of the number of different roots of $a(x)$, $b(x)$ and $c(x)$, the sets of zeros of the derivatives of the functions must have a non-zero overlap that decreases to zero if the set of combined roots of $a^{(m)}(x)$, $b^{(m)}(x)$ and $c^{(m)}(x)$ is maximal for some m.*

Proof. When each of the roots is different,

$$\begin{aligned} &\frac{1}{3}\#\textit{ roots of } (a(x)b(x)c(x)) \\ &\quad \le \max(\deg\ a(x),\ \deg\ b(x),\ \deg\ c(x)) \\ &\quad < \#\textit{ roots of } (a(x)b(x)c(x)). \end{aligned} \tag{2.3}$$

with equality in the first relation occurring only if the degrees of $a(x)$, $b(x)$ and $c(x)$ are equal and the roots are not equal. However, coincidence of any of the roots would be sufficient to transform the relation to an inequality. For the derivatives of the equation,

$$\begin{aligned}
&\frac{1}{3}\#\textit{roots of } a^{(m)}(x)b^{(m)}(x)c^{(m)}(x)) \\
&\quad \leq \max(\deg\ a^{(m)}(x),\ \deg\ b^{(m)}(x),\ c^{(m)}(x)) \\
&\quad < \ \#\ \textit{roots of } (a^{(m)}(x)b^{(m)}(x)c^{(m)}(x))
\end{aligned} \tag{2.4}$$

When the decrease in each of the values of the sums of the number of roots of $a^{(m)}(x)$, $b^{(m)}(x)$ and $c^{(m)}(x)$ is less than three as m increases, the difference between the upper bound and $\max(\deg\ a^{(m)}(x),\ \deg\ b^{(m)}(x),\ \deg\ c^{(m)}(x))$ is reduced as m increases. Furthermore, the lower bound can be saturated for $(a^{(m)}(x), b^{(m)}(x), c^{(m)}(x))$ for some m under these conditions when $\deg\ a(x) = \deg\ b(x) = \deg\ c(x)$. □

Let the roots of $a(x)$ be $a_{01}, ..., a_{0n}$ and the roots of $a'(x)$ be $a_{11}, ...,\ a_{1,n-1}$. Then the convex hull of a_{11},...,$a_{1,n-1}$ is contained in the convex hull of $a_{01}, ..., a_{0n}$. Similarly, the convex hull of the roots of $b'(x)$ is included inside the convex hull of the roots of $b(x)$.

Several possibilities arise for the convex hull of the roots of $c(x)$. Suppose that the convex hull of the zeros of $b(x)$ is located inside the convex hull for $a(x)$. Then this convex region will contain the zeros of $a'(x)$ and $b'(x)$. Any coincident roots of $a'(x)$ and $b'(x)$ would be a zero of $c'(x)$ located inside the domain. The conditions under which the roots of $c(x)$ and $c'(x)$ belong to the convex hull of $a(x)$ may be determined. If the convex hulls of $a(x)$ and $b(x)$ are separate, the convex hull of the entire set of roots is larger than the union of the two different regions. Again, it remains to be established if there is a class of polynomials $c(x)$, equal to the sum of $a(x)$ and $b(x)$, such that the zeros are located in the larger convex hull. Supposing, for example, that $c_\lambda(x) = \lambda a(x) + (1-\lambda)b(x)$, the roots of $c_\lambda(x)$ could be determined as a function of λ, with $c_0(x) = b(x)$, $c_{\frac{1}{2}}(x) = \frac{1}{2}c(x)$ and $c_1(x) = a(x)$. There would be zeros of $c(x)$ located in the region between distinct zeros of $a(x)$ and $b(x)$, and the convex hull could be described by connecting these remaining vertices.

3 Relation to the abc Conjecture

Suppose that there are three integers a, b and c such that $a + b = c$. Let $N(a, b, c)$ be the product of the square-free factors of these integers. There would exist a constant $\kappa_\epsilon > 0$, for any $\epsilon > 0$ such that $c \leq \kappa_\epsilon N(a, b, c)^{1+\epsilon}$ by the abc conjecture [8, 9]. When the three integers are coprime, this product is equal to $\prod_{p_i|abc} p_i$.

Various conclusions may be drawn with respect to integers satisfying the generalized Fermat equation [5] and the polynomial generalization. Suppose, for example, that $A^X + B^Y = C^Z$, $\gcd(A, B, C) > 1$, $X, Y, Z > 2$. Then $N(A^X, B^Y, C^Z) \leq ABC$. Let $\max(A, B, C) = A$ such that $A^X \leq \kappa_\epsilon A^{3+3\epsilon}$ by the abc conjecture and

$$X \leq \frac{\log\ \kappa_\epsilon}{\log\ A} + 3 + 3\epsilon \tag{3.1}$$

When ϵ is set equal to $\frac{1}{83}$, it can be shown that $C \leq \kappa_\epsilon^{83}$ and the number of coprime solutions with and $\frac{1}{X} + \frac{1}{Y} + \frac{1}{Z} < 1$ would be bounded [5, 10].

Theorem 2. *The abc conjecture follows from the inequality for the product of square-free factors of polynomials satisfying a generalized Fermat equation and the specialization to integer values.*

Proof. Consider coprime positive real-valued polynomials in $\mathbb{Q}[x]$ with irreducible factors having values greater than or equal to one such that $a(x)^X + b(x)^Y = c(x)^Z$. Then the product of square-free factors is

$$N(a(x), b(x), c(x)) = \prod_{p_i(x)\|a(x)b(x)c(x)} p_i(x) \leq a(x)b(x)c(x) \tag{3.2}$$

By the analogue of the abc conjecture for polynomials [5],

$$c(x) \leq \kappa_\epsilon (a(x)b(x)c(x))^{1+\epsilon}. \tag{3.3}$$

This bound can be satisfied by a positive polynomial only if the degree Z is less than or equal to three. Setting $a(x)$, $b(x)$ and $c(x)$ equal to constants reduces the equation to that of the abc conjecture. Evidently, no solution with degrees larger than 3 is provided by this method, which is consistent with the conjecture for the equation for coprime polynomials [11, 12].

When $X = Y = Z = 1$, there may exist coprime polynomials such that $a(x) + b(x) = c(x)$. The existence of coprime polynomials $a(x),\ b(x)$ and $c(x)$ with values equal to the coprime integers $a,\ b$ and c follows from the expressions

$$\begin{aligned} a(x) &= a(x_0) + a_1(x - x_0) + ... + a_n(x - x_0)^n \\ b(x) &= b(x_0) + b_1(x - x_0) + ... + b_n(x - x_0)^n \\ c(x) &= c(x_0) + c_1(x - x_0) + ... + c_n(x - x_0)^n \end{aligned} \tag{3.4}$$

where $a = a(x_0)$, $b = b(x_0)$, $c = c(x_0)$ and $c_i = a_i + b_i$, $i = 1, ..., n$. Setting $y = x - x_0$, such that

$$\begin{aligned} a(y) &= a + a_1 y + ... + a_n y^n \\ b(y) &= b + b_1 y + ... + b_n y^n \\ c(y) &= c + c_1 y + ... + c_n y^n, \end{aligned} \tag{3.5}$$

if $k(x) = k + k_1 y + ... + k_m y^m \in \mathbb{Z}$ divides $a(y)$, $b(y)$ and $c(y)$, $k = 1$. Then

$$\begin{aligned} a(y) &= k(y)r(y) = (1 + k_1 y + ... + k_m y^m)(a + r_1 y + ... + r_{n-m} y^{n-m}) \\ b(y) &= k(y)s(y) = (1 + k_1 y + ... + k_m y^m)(b + s_1 y + ... + s_{n-m} y^{n-m}) \\ c(y) &= k(y)t(y) = (1 + k_1 y + ... + k_m y^m)(c + t_1 y + ... + t_{n-m} y^{n-m}) \end{aligned} \tag{3.6}$$

would require

$$\begin{aligned} a_n &= k_m r_{n-m} \\ &\ \vdots \\ a_2 &= k_2 a + k_1 r_1 + r_2 \\ a_1 &= k_1 a + r_1 \\ b_n &= k_m s_{n-m} \\ &\ \vdots \\ b_2 &= k_2 b + k_1 s_1 + s_2 \\ b_1 &= k_1 b + s_1 \end{aligned} \tag{3.7}$$

$$c_n = k_m t_{n-m}$$
$$\vdots$$
$$c_2 = k_2 c + k_1 t_1 + t_2$$
$$c_1 = k_1 c + t_1.$$

There are $3n + n$ relations for $3(n-m) + n + 3n$ coefficients. With $2n - 2m$ free parameters and $3n$ coefficients, three polynomials can be found such that these conditions are not satisfied. Then $\gcd(a(x), b(x), c(x)) = 1$.

More generally, when there may be common factors of these polynomials, let

$$N(a(x), b(x), c(x)) = \prod_{\substack{p_i(x) \| a(x)b(x)c(x) \\ p_i(x_0) = p_i}} p_i(x) \tag{3.8}$$

where p_i is a prime, and

$$N(a(x_0), b(x_0), c(x_0)) = \prod_{p_i \| a(x_0)b(x_0)c(x_0)} p_i. \tag{3.9}$$

The abc conjecture would be a consequence of the more general inequality

$$c(x) \leq \kappa_\epsilon N(a(x), b(x), c(x))^{1+\epsilon} \tag{3.10}$$

for all $\epsilon > 0$ and some constant $\kappa_\epsilon > 0$. It is consistent with the inequality for the degrees of polynomials and excludes exceptional counterexamples for the specialization to integers through the new exponent, which arise from the existence of a minimal set of prime divisors of a large power of an integer. The additional factor $1 + \epsilon$ yields a valid logarithmic inequality since terms of lesser order cannot then cannot have a greater magnitude than the large prime divisor raised to the power $1 + \epsilon$. Consider a number of the form p^{q-1} where p and q are primes. Then $p^{\varphi(q^j)} = p^{q^j - q^{j-1}} \equiv 1\, (mod\, q^j)$ and $1 + n_j q^j = p^{q^j - q^{j-1}}$ for some value of n_j. Since $\ln\, c = (q^j - q^{j-1}) \ln\, p$, $\ln(N(a,b,c)) = \ln(npq) \leq \ln(pq) + \ln\left(\frac{p^{q^j - q^{j-1}}}{q^j}\right) = \ln(pq) + (q^j - q^{j-1}) \ln\, p - \ln\, q^j = \ln\, p + (q^j - q^{j-1}) \ln\, p - (j-1) \ln\, q$. The additional factor $1+\epsilon$ yields a valid logarithmic inequality since terms of lesser order cannot then cannot have a greater magnitude than the large prime divisor raised to the power $1+\epsilon$ in $(q^j - q^{j-1}) \ln\, p \leq C_\epsilon + (1+\epsilon)[\ln\, p + (q^j - q^{j-1}) \ln\, p - (j-1) \ln\, q]$. This conclusion continues to hold for arbitrary bases a and exponents of the form t^k. It has been found also that the ratio $\frac{\log\ c}{\log\ N(a,b,c)}$ for known coprime triples with $a + b = c$ has a maximum value of nearly 1.62991 [13]. Consequently, for the coprime triples, ϵ can be chosen to be a finite number ϵ_0, such that inequality of the abc conjecture is valid trivially for all $\epsilon \geq \epsilon_0$ and holds for $0 < \epsilon < \epsilon_0$ with a sufficiently large constant $C_\epsilon = \log\ \kappa_\epsilon$. The exceptional triples with larger values of $\frac{\log\ c}{\log\ N(a,b,c)}$, which do not belong to one of the earlier types of infinite sequences, would satisfy an inequality with $\epsilon = 0$ and sufficiently large C. Therefore, these triples can be expected to be recovered from the initial inequality for polynomials specialized to integer values. Consequently, each of the coprime integer triples satisfying $a + b = c$ satisfies an inequality consistent with Eq.(3.10).

The reduction of the sum of degrees $a(x)$, $b(x)$ and $c(x)$ arises from multiple or coincident roots of the three polynomials. Given that the product has the form $\alpha(x-\lambda_0)...(x-\lambda_n)$, with the product defined over only the set I of separate roots

$$\prod_{i\in I}\lambda_i \sim \prod_{p_i\|a(x_0)b(x_0)c(x_0)} p_i, \tag{3.11}$$

which is a relation indicating that there exists a power of the form $1+\epsilon$ yielding an equality. Suppose that $c(x)$ has maximal degree and $3\ \deg\ c(x) \geq \#\{i \in I\}$. Then

$$\prod_{p_i\|a(x_0)b(x_0)c(x_0)} p_i \leq \mu_\epsilon \left(\max\ c(x)|_{[\ell_0,\ell_1]|x_0\in[\ell_0,\ell_1]}\right)^{3(1+\epsilon)}, \tag{3.12}$$

which represents a converse to the abc conjecture. However, a reverse inequality would exist by Eq.(2.3). Let the polynomials be positive-semi-definite with zeros only at λ_0,...,λ_n. It follows that

$$\left(c(x)_{inf}|_{[\ell_0,\ell_1]|x_0\in[\ell_0,\ell_1]}\right) \leq \rho_\epsilon \left(\prod_{p_i\|a(x_0)b(x_0)c(x_0)} p_i\right)^{1+\epsilon}, \tag{3.13}$$

where $c(x)_{inf}$ has a value in the vicinity of $c(x_0)$ and it is defined over the region including all of the zeros of $a(x)$, $b(x)$ and $c(x)$. Then, there is a coefficient κ_ϵ such that

$$\frac{\kappa_\epsilon}{\rho_\epsilon} = \frac{c(x_0)}{c(x)_{inf}} \tag{3.14}$$

and

$$c(x_0) \leq \kappa_\epsilon \left(\prod_{p_i\|a(x_0)b(x_0)c(x_0)} p_i\right)^{1+\epsilon}. \tag{3.15}$$

□

4 Conclusion

The degrees of solutions to the equation $a(x)+b(x) = c(x)$ are known to satisfy an inequality depending on the common zeros of the polynomials. A common zero represents a nontrivial factor, and therefore, this equation can be reduced to proper form such that the polynomials have a greatest common divisor equal to one. This result is a special case of the reduction of the generalized Fermat equation when $X = Y = Z$.

It is established that the product of square-free factors of these polynomials must be less than the product of the positive polynomials $a(x)$, $b(x)$ and $c(x)$. A direct analogue of the inequality $\max(\deg\ a(x),\ \deg\ b(x),\ \deg\ c(x)) \leq \deg(rad(a(x)b(x)c(x)) - 1$ for coprime polynomials is known to be not valid for integers and must be modified with the exponent $1+\epsilon$ and a constant $C_\epsilon = \log\ \kappa_\epsilon$ such that it would be $\log\ c \leq C_\epsilon + (1+\epsilon)\log(N(abc))$ [9]. Therefore, adjusting the inequality for polynomials, which would remain valid for $a(x)$, $b(x)$ and $c(x)$ satisfying $a(x)+b(x) = c(x)$, specialization to prime factors of the product of the values of the polynomials would yield a result equivalent to the abc conjecture.

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