Extensions of D’Aurizio’s trigonometric inequality

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Received: 15 October 2016
Accepted: 31 March 2017

Abstract: We offer extensions of D’Aurizio’s trigonometric inequality, as well to its counterpart, proved in [1] and [2].
Keywords: Inequalities, Trigonometric functions, D’Aurizio’s inequality.
AMS Classification: 26D15, 26D99.

1 Introduction

D’Aurizio’s trigonometric inequality states that (see [1, 3]) for any \( x \in (0, \frac{\pi}{2}) \) one has

\[
\frac{1 - \frac{\cos x}{\cos \left( \frac{x}{2} \right)}}{x^2} < \frac{4}{\pi^2}.
\]  

(1)

D’Aurizio’s proof was based on certain infinite product expansions, as well as inequalities on infinite series and Riemann’s zeta function. The author ([3]) has obtained a new proof, by applying known trigonometric inequalities and an auxiliary function. The method implied also the following counterpart to (1):

\[
\frac{3}{8} < \frac{1 - \frac{\cos x}{\cos \left( \frac{x}{2} \right)}}{x^2}
\]  

(2)

for any \( x \in (0, \frac{\pi}{2}) \).

Our aim in what follows is to extend inequalities (1) and (2) for any positive integer \( n \), in place of 2.
For example, for \( n = 3 \) we have the following double inequality:
\[
\frac{4}{\pi^2} < \frac{1 - \frac{\cos x}{\cos(x/3)}}{x^2} < \frac{4}{9}.
\]

\section{Main results}

The following extensions of inequalities (1)–(3) will be proved.

**Theorem 1.** For any \( x \in (0, \frac{\pi}{2}) \) and all positive integers \( n \geq 3 \), one has
\[
\frac{4}{\pi^2} < \frac{1 - \frac{\cos x}{\cos(x/n)}}{x^2} < \frac{n^2 - 1}{2n^2}.
\]

**Proof.** As in [3], let \( x = nt \), where
\[
f(x) = \frac{1 - \frac{\cos x}{\cos(x/n)}}{x^2}.
\]

Put
\[
g(t) = \cos t - \cos nt \quad \frac{t^2 \cos t}{n^2 \cos t},
\]
so clearly
\[
g(t) = n^2 \cdot f(nt).
\]

In order to study the monotonicity of function \( f(x) \) upon \( x \), it will be sufficient to consider the monotonicity of \( g(t) \) upon \( t = \frac{x}{n} \in (0, \frac{\pi}{2n}) \). It was shown in [3] that for \( n = 2 \), \( g(t) \) is strictly increasing. Now let us see the case \( n = 3 \). In this case one has
\[
g(t) = \frac{\cos t - 4(\cos t)^3 + 3 \cos t}{t^2 \cos t} = 4 \frac{1 - (\cos t)^2}{t^2} = 4 \left( \frac{\sin t}{t} \right)^2.
\]

As the function \( s(t) = \frac{\sin t}{t} \) is known to be strictly decreasing on \((0, \frac{\pi}{2})\) (see e.g. [2]), it will be strictly decreasing also on \((0, \frac{\pi}{3})\), so we get that for \( n = 3 \), the function \( g(t) \) introduced above is strictly decreasing, contrary to the case \( n = 2 \). This implies immediately inequalities (3).

Now, we shall prove that, in the general case, for any \( n \geq 3 \), \( g(t) \) is strictly decreasing function. First computing the derivative of function \( g(t) \), one obtains
\[
(t^3).g'(t) = \frac{2 \cos nt. \cos t - 2(\cos t)^2 + nt. \sin nt. \cos t - t \cos nt. \sin t}{(\cos t)^2} = h(t).
\]

In order to prove that \( g'(t) < 0 \), it will be sufficient to show that \( h(t) < 0 \). One has \( h(0) = 0 \), so it will be enough to prove that \( h'(t) < 0 \). By using (5), and the classical addition formulae
\[
\sin(a + b) = \sin a. \cos b + \cos a. \sin b,
\]
\[
\sin(a - b) = \sin a. \cos b - \cos a. \sin b,
\]
after some quite long and elementary computations (which we omit here), the following can be deduced: \( h'(0) = 0 \) and for the second derivative of \( h \) one has:
\[
[8n^2.(\cos t)^4].h''(t)
\]
= [(n + 1)^3 \cdot \sin(n - 3)t + (n - 1)^3 \cdot \sin(n + 3)t + A(n) \cdot \sin(n - 1)t + B(n) \cdot \sin(n + 1)t], \quad (6)

where

\[ A(n) = 3n^3 + 3n^2 - 15n - 23 \]

and

\[ B(n) = 3n^3 - 3n^2 - 15n + 23. \]

Now, remark that \( A(n) = 3n(n^2 - 5) + 3n^2 - 23 > 0 \), as for \( n \geq 3 \) we get \( n^2 - 5 \geq 4 > 0 \) and \( 3n^3 - 23 \geq 4 > 0 \). Similarly, \( B(n) = 3n(n^2 - n - 5) + 23 \), with \( n^2 - n - 5 = n(n - 1) - 5 \geq 6 - 5 = 1 \), so we get \( B(n) > 0 \) again.

As \((n - 1)t\) is in \((0, \frac{n - 1}{n} \pi)\), which is in \((0, \frac{\pi}{2})\), and similarly for \((n - 3)t\) for \( n > 3 \) (for \( n = 3 \), one has \((n - 3)t = 0\)), by relation (6) we get that \( h''(t) < 0 \). This implies \( h'(t) < h'(0) = 0 \), so that \( h(t) < h(0) = 0 \), giving that \( g(t) \) is strictly decreasing. Thus the function \( f \) is strictly decreasing, too. Finally, inequalities (4) are consequences of the monotonicity of \( f \), implying: \( f\left(\frac{\pi}{2} - \right) < f(x) < f(0+) \), and using the L’Hôpital rule.

\[ \square \]

**Remark 1.** As it is well-known that \( \cos(nt) = T_n(\cos t) \), where \( T_n \) are the classical Chebyshev polynomials, we get from the above proved results, that the fraction \( \cos t - \frac{T_n(\cos t)}{t^2 \cos t} \) is a strictly decreasing function of \( t \) for any \( n \geq 3 \), while for \( n = 2 \) it is strictly increasing.

**Conjecture 1.** We conjecture that, the above function \( g(t) \) is strictly decreasing for any real number \( n \geq 3 \).

**References**

