On two arithmetic functions

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Abstract: Some properties of two new arithmetic functions are studied. Three conjectures are formulated.

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1 Introduction

In a recent paper [1], it was introduced the following arithmetic function: let \( p \) be a prime number, and let \( \downarrow p \) denote the greatest prime smaller than \( p \), for \( p \geq 3 \), and let \( \downarrow p = 1 \), if \( p = 2 \). If \( n \geq 2 \) has the prime factorization \( n = \prod_{i=1}^{r} p_i^{\alpha_i} \), where \( k, \alpha_1, ..., \alpha_r, r \geq 1 \) are natural numbers and \( p_1, ..., p_r \) are different primes, then let us define

\[
\downarrow n = \prod_{i=1}^{r} (\downarrow p_i)^{\alpha_i}.
\]  

(1)

Let \( \downarrow 1 = 1 \).
It is immediate that for distinct primes $p$ and $q$ one has $\downarrow (pq) = \downarrow p \cdot \downarrow q$, so by (1) it is immediate that for $(n,m) = 1$ one has $\downarrow (nm) = \downarrow n \cdot \downarrow m$, which means that this arithmetical function is multiplicative function.

In what follows we define the dual of this arithmetic function as follows. For a prime $p$, let $\uparrow p$ denote the least prime greater than $p$. For example, $\uparrow 2 = 3$, $\uparrow 3 = 5$, etc. Similarly to (1), we define for $n \geq 2$:

$$\uparrow n = \prod_{i=1}^{r}(\uparrow p_i)^{\alpha_i}. \hspace{1cm} (2)$$

Let $\uparrow 1 = 1$. Then this arithmetical function is multiplicative, too.

## 2 Main results

**Lemma 1.** One has $\downarrow p \leq p - 2$ for any $p \geq 5$ \hspace{1cm} (3) and $\uparrow p \geq p + 2$ for any $p \geq 3$. \hspace{1cm} (4)

**Proof.** For $p$ odd prime, $p - 1$ is even number, and this is not prime for $p - 1$ distinct from 2. Thus the greatest prime $q < p$ will be in the best possible case, the number $q = p - 2$. These are not valid for $p = 2, 3$, but for $p \geq 5$, are true, so (3) follows. The proof of (5) follows on the same lines.

Obviously, one has $\downarrow p = p - 1$ for $p = 2$ or $p = 3$ \hspace{1cm} (5) and $\uparrow p = p + 1$ for $p = 2$ \hspace{1cm} (6)

We see directly, that there is equality in (3) if and only if the pair $(p - 2, p)$ is a twin pair. Similarly, there is equality in (4) if and only if the pair $(p, p + 2)$ is a twin pair.

It is not known, if there exist infinitely many such pairs, and this is one of the most notorious open problems of Number theory.

**Lemma 2.** One has $\uparrow p - \downarrow p \geq 6$ for any prime $p \geq 7$. \hspace{1cm} (7)

**Proof.** By Lemma 1, one has $\uparrow p - \downarrow p \geq 4$ for any $p \geq 5$. We shall prove that for a prime $p \geq 7$, no all terms of the three numbers $p - 2, p, p + 2$ cannot be primes. This is true however for $p = 5$. It is well-known that any prime $p \geq 5$ can be written in one of the following forms: $p = 6k - 1$ or $p = 6k + 1$. In the first case, one has $p - 2 = 6k - 3$, divisible by 3, and so not prime for $k \geq 2$. In the second case $p + 2 = 6k + 3$ is divisible again by 3, and is not prime for $k \geq 1$. These prove essentially inequality (7).

Obviously, one has $\uparrow p - \downarrow p = 2$ for $p = 2$. \hspace{1cm}
\[ \uparrow p - \downarrow p = 3 \text{ for } p = 3, \]
\[ \uparrow p - \downarrow p = 4 \text{ for } p = 4. \]

This means that one has
\[ \uparrow p - \downarrow p \geq 2 \text{ for any } p \geq 2. \]

\[ \textbf{Lemma 3.} \text{ Let } \{x_i\}_{i=1}^r \text{ and } \{y_i\}_{i=1}^r \text{ be two sequences of positive real numbers. Then one has} \]
\[ (x_1 + y_1) \cdots (x_r + y_r) \geq x_1 \cdots x_r + y_1 \cdots y_r. \]

\[ \text{If } x_i - y_i \geq 1, \text{ then} \]
\[ (x_1 - y_1) \cdots (x_r - y_r) \leq x_1 \cdots x_r - y_1 \cdots y_r. \]

\[ \textbf{Proof.} \text{ (8) is well-known, and follows, e.g., on induction upon } r. \text{ For the proof of (9) apply (8) for } x_i - y_i \text{ instead of } x_i \text{ and } y_i \text{ for } y_i. \text{ Then (8) becomes (9).} \]

\[ \textbf{Theorem 2.1.} \text{ One has} \]
\[ \uparrow n \geq n + 2^{\Omega(n)} \text{ for any odd } n \geq 3 \]
\[ \text{and} \]
\[ \downarrow n \leq n - 2^{\Omega(n)} \text{ for any } n \text{ not divisible by 6}, \]

where \( \Omega(n) \) denotes the total number of prime factors of \( n \) (i.e. for the prime factorization of \( n \) in the Introduction, \( \Omega(n) = \sum_{i=1}^r a_i \)).

\[ \textbf{Proof.} \text{ By relation (4) of Lemma 1, and by Lemma 3 , relation (8), one can write} \]
\[ \uparrow n \geq (p_1 + 2)^{a_1} \cdots (p_r + 2)^{a_r} \geq (p_1^{a_1} + 2^{a_1}) \cdots (p_r^{a_r} + 2^{a_r}) \geq n + 2^{a_1 + \cdots + a_r} = n + 2^{\Omega(n)}, \]

which proves (10). The proof of (11) goes on the same lines, by using relation (5) of Lemma 1 and relation (9) of Lemma 3.

Now, we extend relation (7).

\[ \textbf{Theorem 2.2.} \text{ One has} \]
\[ \uparrow n - \downarrow n \geq 2^{\Omega(n)}, \text{ for any } n \geq 2. \]

\[ \textbf{Proof.} \text{ Actually, we will prove a stronger result , by using the following inequality of H"older.} \]

\[ \textbf{Lemma 4.} \text{ If } \{x_i\}_{i=1}^r \text{ and } \{y_i\}_{i=1}^r \text{ be two sequences of positive real numbers. Then one has} \]
\[ ((x_1 + y_1) \cdots (x_r + y_r))^{1/r} \geq (x_1 \cdots x_r)^{1/r} + (y_1 \cdots y_r)^{1/r}. \]

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Apply now inequality (7), and (13) for $x_i = 2^{a_i}$, $y_i = (\downarrow p_i)^{a_i}$. Then we get

$$\uparrow n \geq (2^{\Omega(n)/\omega(n)}) + (\downarrow n)^{1/\omega(n)\omega(n)},$$

where $r = \omega(n)$ denotes the number of distinct prime factors of $n$.

It is immediate now that (14) is a refinement of (12), which follows at once from the inequality $(a + b)^r \geq a^r + b^r$, with $a$ the first term, while $b$ is the second term in the parentheses of (14). Therefore, (14) follows, even in the improved form (12).

The following limit properties are valid:

**Theorem 2.3.** One has

$$\lim_{p \to \infty} \frac{\downarrow p}{p} = \lim_{p \to \infty} \frac{\uparrow p}{p} = \lim_{p \to \infty} \frac{\downarrow p}{\uparrow p} = 1,$$

$$\lim_{n \to \infty} \frac{\downarrow n}{n} = 0, \lim_{n \to \infty} \frac{\uparrow n}{n} = 1,$$

$$\lim_{n \to \infty} \frac{\uparrow n}{n} = 1, \lim_{n \to \infty} \frac{\uparrow n}{n} = \infty.$$

**Proof.** Let $p_1 < p_2 < \cdots < p_{k-1} < p_k < p_{k+1} < \cdots$ be the increasing sequence of the consecutive primes, and suppose that $p = p_k$. Then one has $\downarrow p = p_{k-1}$ and $\uparrow p = p_{k+1}$. Therefore, relation (15) becomes

$$\lim_{k \to \infty} \frac{p_{k-1}}{p_k} = \lim_{k \to \infty} \frac{p_{k+1}}{p_k} = \lim_{k \to \infty} \frac{p_{k-1}}{p_{k+1}} = 1.$$

The first two relations are well-known (see e.g. [3]), and are in fact consequences of the prime number theorem written in the form

$$\lim_{k \to \infty} \frac{p_k}{k \log k} = 1.$$

The last relation of (18) follows by the identity

$$\frac{p_{k-1}}{p_{k+1}} = \frac{p_{k-1}}{p_k} \frac{p_k}{p_{k+1}},$$

and the first two relations. For the proof of the first relation of (16) it is sufficient to consider the sequence of numbers $n = 3^k$. Then

$$\downarrow n = \left(\frac{2}{3}\right)^k,$$

which tends to zero, as $k$ tends to infinity.

For the second relation of (16) remark that $\downarrow n \leq 1$, and for the particular case $n = p$ (prime), by (15) the limit is 1. Therefore the $\limsup$ should be equal to 1.

The first equality of (17) follows on the same lines, by remarking that $\downarrow n \geq 1$, and using again (15). For the second relation, let us take again $n = 3^k$, when

$$\downarrow n = \left(\frac{5}{3}\right)^k,$$

which tends to infinity for $k$ tending to $\infty$.

It is immediate consequence of (15) and (17) that

$$\lim \inf_{n \to \infty} \frac{\uparrow n}{\downarrow n} = 1 \text{ and } \lim \sup_{n \to \infty} \frac{\uparrow n}{\downarrow n} = \infty.$$
By a particular case of a theorem of Maynard ([2]) one gets that

$$\liminf_{n \to \infty} (p_{k+1} - p_k) \leq C,$$

where $C > 0$ is a constant. This implies immediately:

Theorem 2.4. One has

$$\liminf_{p \to \infty} \frac{\uparrow p - \downarrow p}{\log p} = 0. \quad (19)$$

One has

$$\liminf_{n \to \infty} \frac{\uparrow n - \downarrow n}{\log n} = 0 \text{ and } \limsup_{n \to \infty} \frac{\uparrow n - \downarrow n}{\log n} = \infty. \quad (20)$$

The first relation of (20) is a consequence of (19), while the second relation follows by the remark that

$$\frac{p_{k+1} - p_k - 1}{\log p_k} > \frac{p_{k+1} - p_k}{\log p_k} = w_k$$

and it is well-known by a result of Westzynthius (see, [3]) that $\limsup w_k = \infty$.

Suggested by Lemma 2, we formulate

Conjecture 1. For each prime number $p$:

$$\liminf_{p \to \infty} (\uparrow p - \downarrow p) = 6.$$

For the prime number $p$ let us define

$$\Delta(p) = \frac{\uparrow p + \downarrow p}{2},$$

$$E(p) = \frac{\uparrow p - \downarrow p}{2},$$

$$Z(p) = |p - \Delta(p)|.$$

Let us construct the following Table.

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**Conjecture 2.** For each prime number $p$:

$$\lfloor \ln p \rfloor \leq \max_{q \leq p} E(q).$$

**Conjecture 3.** For each prime number $p$:

$$\lfloor \ln \ln p \rfloor \leq \max_{q \leq p} Z(q).$$

**References**

