The $q$-Lah numbers
and the $n$-th $q$-derivative of $\exp_q\left(\frac{1}{x}\right)$

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Abstract: A recently reported nice and surprising property of the Lah numbers is shown to hold for the $q$-Lah numbers as well, i.e., they can be obtained by taking successive $q$-derivatives of $\exp_q\left(\frac{1}{x}\right)$, where $\exp_q(x)$ is the $q$-exponential.

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1 Introduction

The Lah numbers are the transformation coefficients allowing the expression of a rising factorial as a linear combination of falling factorials, i.e.,

$$x(x + 1)(x + 2) \cdots (x + k - 1) = \sum_{\ell = 1}^{k} L(k, \ell) x(x - 1)(x - 2) \cdots (x - \ell + 1).$$

The Lah numbers are given by the simple explicit expression

$$L(k, \ell) = \binom{k}{\ell} \frac{(k - 1)!}{(\ell - 1)!}.$$  

They are closely related to the Stirling numbers, satisfying

$$L(k, \ell) = \sum_{j=\ell}^{k} \left[j \atop \ell\right] \left\{j\atop k\right\},$$

and they satisfy the recurrence relation

\[ L(k+1, \ell) = (k+\ell)L(k, \ell) + L(k, \ell-1), \]

with \( L(1, 1) = 1 \).

It was recently shown by Daboul et al. [1] that the \( k \)-th derivative of \( \exp \left( \frac{1}{x} \right) \) yields the Lah numbers, i.e.,

\[
\frac{d^k}{dx^k} \left( \exp \left( \frac{1}{x} \right) \right) = (-1)^k \exp \left( \frac{1}{x} \right) \sum_{\ell=1}^{k} L(k, \ell) \frac{x^{k+\ell}}{x^{k+\ell}}. \tag{1}
\]

The \( q \)-Lah numbers

\[
[n]_q[n+1]_q \cdots [n+k-1]_q = \sum_{\ell=1}^{k} L_q(k, \ell)[n]_q[n-1]_q \cdots [n-\ell+1]_q, \tag{2}
\]

were introduced by Garsia and Remmel [2], who derived the recurrence relation

\[
L_q(k+1, \ell) = [k+\ell]_q L_q(k, \ell) + q^{k+\ell-1} L_q(k, \ell-1) \tag{3}
\]

and the explicit expression

\[
L_q(k, \ell) = \binom{k}{\ell} \frac{[k-1]_q!}{[\ell-1]_q! q^{\ell-1} \ell}. \]

In the next section we derive the \( q \)-analogue of equation (1).

## 2 \( q \)-analogue of equation (1)

Recall the definitions of the \( q \)-exponential

\[
\exp_q(x) = \sum_{i=0}^{\infty} \frac{x^i}{[i]_q!},
\]

of the \( q \)-derivative

\[
Df(x) = \frac{f(qx) - f(x)}{x(q-1)},
\]

and the \( q \)-Leibniz rule for the \( q \)-derivative of a product

\[
D \left( f(x)g(x) \right) = (Df(x))g(qx) + f(x)(Dg(x)).
\]

The following relations are easily established

\[
D \left( \frac{1}{(xq^k-1)^\ell} \right) = -q^k \frac{[\ell]_q}{(xq^k)^{\ell+1}},
\]

and

\[
D \exp_q \left( \frac{1}{xq^k-1} \right) = -q^k \frac{1}{(xq^k)^2} \exp_q \left( \frac{1}{xq^k} \right). \tag{4}
\]
Hence,

\[ D \left( \frac{1}{x q^{k-1}} \right) \exp_q \left( \frac{1}{x e^{k-1}} \right) = -q^k \left( \frac{[\ell]_q}{x q^k \ell + 1} + \frac{q^\ell}{x q^k \ell + 2} \right) \exp_q \left( \frac{1}{x q^k} \right). \quad (5) \]

The expression for the \( q \)-derivative of \( \exp_q \left( \frac{1}{x} \right) \) yields a finite sum involving the \( q \)-Lah numbers \( L_q(k, \ell) \), defined by equation (2), as stated in

**Theorem 1.**

\[ D^k \left( \exp_q \left( \frac{1}{x} \right) \right) = (-1)^k q^{\binom{k+1}{2}} \exp_q \left( \frac{1}{x^k} \right) \sum_{\ell=1}^{k} \frac{L_q(k, \ell)}{x^\ell} \]

where \( x_k = x q^k \).

**Proof.** By induction. For \( k = 1 \) we obtain \( D \left( \exp_q \left( \frac{1}{x} \right) \right) = -q \exp_q \left( \frac{1}{x^2} \right) \frac{L_q(1,1)}{(x^2 q^2)} \), which is consistent with equation (4), since \( L_q(1, 1) = 1 \).

Now, assume that the theorem holds for \( k \) and take the \( q \)-derivatives of both sides of equation (6). The left-hand-side becomes

\[ D^{k+1} \left( \exp_q \left( \frac{1}{x} \right) \right) = (-1)^{k+1} q^{\binom{k+2}{2}} \exp_q \left( \frac{1}{x^{k+1}} \right) \sum_{\ell=1}^{k+1} \frac{L_q(k+1, \ell)}{x^\ell} \]

and, using equation (5) followed by an appropriate shift of the summation index, the right-hand side becomes

\[ (-1)^{k+1} q^{\binom{k+2}{2}} \exp_q \left( \frac{1}{x^{k+1}} \right) \sum_{\ell=1}^{k+1} \frac{1}{x^\ell} \left( [k + \ell]_q L_q(k, \ell) + q^{k+\ell-1} L_q(k, \ell - 1) \right). \]

One readily obtains the recurrence relation, equation (3). \( \square \)

Different \( q \)-analogues of the Lah numbers have been considered by Lindsay, Mansour and Shattuck [3] and by Wagner [4]. Whether they can be produced by appropriately modified \( q \)-differentiations and \( q \)-exponentials remains to be seen.

**References**


