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The primitive solutions to the Diophantine equation $2X^4 + Y^4 = Z^3$

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Abstract: We find all primitive non-zero integer solutions to the title equation, namely $(x, y, z) = (\pm 5, \pm 3, 11)$. The proofs involved are based solely on elementary methods with no use of computers and the elliptic curve machinery.
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1 Introduction

Terai and Osada and Cao published two papers in the early 1990s concerning the similar equations $x^4 + dy^4 = z^p$ and $cx^4 + dy^4 = z^p$, where p is an odd prime. They showed that these equations have no integer solutions if certain conditions are fulfilled [1, 2]. According to a theorem of Darmon and Granville, the equation $Ax^p + By^q = Cz^r$ has only a finite number of primitive non-zero solutions (i.e., Ax, By and Cz are pairwise relatively prime and $x \cdot y \cdot z \neq 0$) if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ for A, B, C fixed non-zero integers and p, q, r fixed integers > 1 [3]. Applying this we conclude that the title equation has a finite number of primitive non-zero solutions. The similar equation $2x^2 + y^4 = z^n$ has been examined for all $n \geqq 4$. Combined works using elliptic curves for different exponents (n) and the method of Galois representations and modularity have shown that the only primitive positive non-zero solution to this equation is $(x, y, z, n) = (11, 1, 3, 5)$ [4, 5]. Many equations with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and A, B, C fixed non-zero integers have been completely solved [6, 7, 8], in appropriate cases simply and solely with elementary methods [9]. For the special case when $A = B = C = 1$, i.e., $x^p + y^q = z^r$ there is a conjecture stating that there are no primitive non-zero solutions when min $(p, q, r) \geqq 3$ [7,10]. However, should the abc-conjecture become a theorem there exist only a finite number of primitive non-zero solutions to the equation $Ax^p + By^q = Cz^r$ for A, B, C fixed nonzero integers and all positive p, q, r such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ [11] where solutions arising from the identity $1^p + 2^3 = 3^2$ are excluded.

In this work, we determine all primitive non-zero solutions to the title equation using only elementary methods and at one crucial step with the aid of an old theorem of Lucas [12].

Lemma 1. *The Diophantine equation $24z^4 = x^4 - y^4$ has no non-zero solution.*

Proof. With no loss of generality we may assume that $(24z, x, y) = 1$. Hence $24z, x$ and y are pairwise relatively prime and after congruence considerations we realize that x and y are odd and z is even. Substitute $x = p + q$ and $y = p - q$ where $p \not\equiv q \bmod 2$. $(p, q) = 1$ since $(x, y) = 1$. Hence, $24z^4 = 2^3 \cdot (p^2 + q^2) \cdot p \cdot q \Longrightarrow 3z^4 = (p^2 + q^2) \cdot p \cdot q$. Since $(p, q) = 1$ we see that $3 \nmid (p^2 + q^2)$. Hence $3 \mid p$ or $3 \mid q$. If $3 \mid p$, we have $p = 3v$. Hence $z^4 = (p^2 + q^2) \cdot v \cdot q$. However $p^2 + q^2, v$ and q are pairwise relatively prime and we have $p^2 + q^2 = A^4$ and $q = B^4$. Hence $p^2 + (B^4)^2 = A^4 \Longrightarrow p^2 = A^4 - (B^2)^4$ and this well-known equation has no non-zero solutions and a proof of this fact can be found in e.g. [13]. If $3 \mid q$, we will have a contradiction in the same way. $\square$

Theorem 1. *The only primitive non-zero solutions to the equation $2x^4 + y^4 = z^3$ are $(x, y, z) = (\pm 5, \pm 3, 11)$.*

Proof. From prerequisites we see that $2x, y$ and z are pairwise relatively prime and $x \cdot y \cdot z \neq 0$. Thus, after congruence considerations we realize that y and z must be odd. We get,

$(y^2 + x^2\sqrt{-2}) \cdot (y^2 - x^2\sqrt{-2}) = z^3$ where $y^2 + x^2\sqrt{-2}$ and $y^2 - x^2\sqrt{-2}$ are coprime in $Q(\sqrt{-2})$. Since $Q(\sqrt{-2})$ has unique factorisation and all units (± 1) are cubes we have,

$y^2 + x^2\sqrt{-2} = (a + b\sqrt{-2})^3$. Hence,

$y^2 = a(a^2 - 6b^2)$ and $x^2 = b(3a^2 - 2b^2)$. Since y is odd we see that a must be odd. $(a, b) = 1$ since $(x, y) = 1$.

Case I. $3 \nmid a$ and $3 \nmid b$

Hence $(a, a^2 - 6b^2) = 1$ and $(b, 3a^2 - 2b^2) = 1$. From $y^2 = a(a^2 - 6b^2)$ it follows that

$$a = \pm U_1^2 \tag{1}$$

and

$$a^2 - 6b^2 = \pm U_2^2 \tag{2}$$

where U_1 and U_2 are odd. $(a, U_2) = 1$. The negative sign in (1) and (2) must be rejected after reduction modulo 3 of equation (2). From (1) and (2) we get $6b^2 = U_1^4 - U_2^2$. Hence we must after congruence considerations conclude that b must be even. On the other hand we see from $x^2 = b(3a^2 - 2b^2)$ that

$$b = \pm V^2 \tag{3}$$

and

$$3a^2 - 2b^2 = \pm U_3^2. \tag{4}$$

Since $(b, U_3) = 1$ we must after reduction modulo 3 of equation (4) reject the negative sign in (3) and (4). Thus from equation (4) we have $2b^2 = 3a^2 - U_3^2$ and since a and U_3 are odd we realize after congruence considerations that b must be odd. Hence we have a contradiction and Case I must be impossible.

Case II. $3 \nmid a$ and $3 \mid b$

Hence $b = 3v$. Since $(a, a^2 - 6b^2) = 1$ it follows from $y^2 = a(a^2 - 6b^2)$ that $a = \pm U_1^2$ (1) and $a^2 - 6b^2 = \pm U_3^2$ (2). U_1 and U_3 are odd and $(a, U_3) = 1$. The negative sign is rejected in (1) and (2) after reduction modulo 3 of equation (2). From (1) and (2) we have $U_1^4 - 6b^2 = U_3^2$ (3). With $b = 3v$ (4) we see from $x^2 = b(3a^2 - 2b^2)$ that $3v(3a^2 - 2 \cdot 9v^2) = x^2 \Longrightarrow 9v(a^2 - 6v^2) = x^2$. Since $(9v, a^2 - 6v^2) = 1$ we get

$$v = \pm V^2 \tag{5}$$

and

$$a^2 - 6v^2 = \pm U_2^2 \tag{6}$$

where U_2 is odd. Since $(a, U_2) = 1$ we must reject the negative sign in (5) and (6) after reduction modulo 3 of equation (6). From (3), (4) and (5) we have

$$U_1^4 - 54V^4 = U_3^2 \tag{7}$$

and from (1), (5) and (6) we have

$$U_1^4 - 6V^4 = U_2^2. \tag{8}$$

From (7) and (8) we see that V must be even. Furthermore $(8) - (7) \Longrightarrow$

$$48V^4 = U_2^2 - U_3^2. \tag{9}$$

From (8) we get

$$9U_1^4 - 54V^4 = 9U_2^2. \tag{10}$$

$(10) - (7) \Longrightarrow$

$$8U_1^4 = 9U_2^2 - U_3^2. \tag{11}$$

From (9) we have,

$$48V^4 = (U_2 + U_3) \cdot (U_2 - U_3).$$

Let $V^4 = 2^{4k} \cdot U^4$ where U is odd and $k \geqq 1$. Hence,

$$48V^4 = 3 \cdot 2^{4k+4} \cdot U^4 = (U_2 + U_3) \cdot (U_2 - U_3),$$

where $U_2 + U_3$ and $U_2 - U_3$ can be expressed as $\pm 2p$ and $\pm 2q$, respectively. Moreover $p \not\equiv q \bmod 2$ and $(p, q) = 1$ since $(U_2, U_3) = 1$. Thus we have the following possibilities since $3 \mid p$ or $3 \mid q$,

i1.) $U_2 + U_3 = \pm 2A^4$ and $U_2 - U_3 = \pm 3 \cdot 2^{4k+3} \cdot B^4$. Hence
$U_2 = \pm(A^4 + 3 \cdot 2^{4k+2} \cdot B^4), U_3 = \pm(A^4 - 3 \cdot 2^{4k+2} \cdot B^4)$ and $3 \nmid A$.

i2.) $U_2 + U_3 = \pm 3 \cdot 2A^4$ and $U_2 - U_3 = \pm 2^{4k+3} \cdot B^4$. Hence
$U_2 = \pm(3A^4 + 2^{4k+2} \cdot B^4), U_3 = \pm(3A^4 - 2^{4k+2} \cdot B^4)$ and $3 \nmid B$.

i3.) $U_2 + U_3 = \pm 2^{4k+3} \cdot A^4$ and $U_2 - U_3 = \pm 3 \cdot 2B^4$. Hence
$U_2 = \pm(2^{4k+2} \cdot A^4 + 3B^4), U_3 = \pm(2^{4k+2} \cdot A^4 - 3B^4)$ and $3 \nmid A$.

i4.) $U_2+U_3=\pm 3\cdot 2^{4k+3}\cdot A^4$ and $U_2-U_3=\pm 2B^4$. Hence
$U_2=\pm(3\cdot 2^{4k+2}\cdot A^4+B^4), U_3=\pm(3\cdot 2^{4k+2}\cdot A^4-B^4)$ and $3\nmid B$.
$U=A\cdot B$ where $(A,B)=1$ since $(U_2,U_3)=1$.

From equation (11) we have $8U_1^4=(3U_2+U_3)\cdot(3U_2-U_3)$. According to the previous discussion, we have the following alternatives,

ii1.) $3U_2+U_3=\pm 4C^4$ and $3U_2-U_3=\pm 2D^4$. Hence
$U_2=\pm(\frac{2C^4+D^4}{3})$ and $U_3=\pm(2C^4-D^4)$

ii2.) $3U_2+U_3=\pm 2C^4$ and $3U_2-U_3=\pm 4D^4$. Hence
$U_2=\pm(\frac{C^4+2D^4}{3})$ and $U_3=\pm(C^4-2D^4)$.
$U_1=C\cdot D$ where $(C,D)=1$ and $3\nmid C\cdot D$ since $(3U_2,U_3)=1$.

N.B. Concerning the expressions of U_2 and U_3 in i1.)–i4.) and ii1.)–ii2.) we have $U_2=\pm(.....)$ and $U_3=\pm(.....)$. This certainly means that $U_2=+(.....)$ and $U_3=+(.....)$ or $U_2=-(.....)$ and $U_3=-(.....)$. If solutions exist at least one parametric solution of U_2 and U_3 in i1.)–i4.) must be equal to at least one parametric solution of U_2 and U_3 in ii.1)–ii.2) for some value (values) of A,B,C and D. Thus, we have the following possibilities,

i1.) = ii1.):

$$U_2=\pm(A^4+3\cdot 2^{4k+2}\cdot B^4)=\pm(\frac{2C^4+D^4}{3}). \tag{12}$$

Since A^4,B^4,C^4 and D^4 are positive the signs in (12) are not independent and we have,

$$3A^4+9\cdot 2^{4k+2}\cdot B^4=2C^4+D^4. \tag{13}$$

$$U_3=\pm(A^4-3\cdot 2^{4k+2}\cdot B^4)=\pm\left(2C^4-D^4\right) \tag{14}$$

and since the signs in (12) are not independent so are the signs in (14) and we get

$$A^4-3\cdot 2^{4k+2}\cdot B^4=2C^4-D^4. \tag{15}$$

Equation (13) - (15) $\Longrightarrow$

$$2A^4+12\cdot 2^{4k+2}\cdot B^4=2D^4\Longrightarrow 24(2^k\cdot B)^4=D^4-A^4. \tag{16}$$

However according to Lemma 1 equation (16) has no non-zero solutions and possible zero solutions violate the condition $x\cdot y\cdot z\neq 0$ in the title equation.

i1.) = ii2.):

$$U_2=\pm(A^4+3\cdot 2^{4k+2}\cdot B^4)=\pm\left(\frac{C^4+2D^4}{3}\right) \tag{17}$$

and

$$U_3=\pm(A^4-3\cdot 2^{4k+2}\cdot B^4)=\pm\left(C^4-2D^4\right). \tag{18}$$

In analogy with previous discussion in i1.) = ii1.) we see that the signs in (17) and (18) are not independent. Hence from equation (18) we have $A^4-3\cdot 2^{4k+2}\cdot B^4=C^4-2D^4$ so $2D^4-3\cdot 2^{4k+2}\cdot B^4=C^4-A^4$. Since A,B,C and D are all odd we have $2D^4-3\cdot 2^{4k+2}\cdot B^4\equiv 0 \bmod 16$ and this is impossible.

i2.) $=$ ii1.): $U_2 = \pm(3A^4 + 2^{4k+2} \cdot B^4) = \pm\left(\frac{2C^4+D^4}{3}\right)$ and since the signs are not independent we have $9A^4 + 3 \cdot 2^{4k+2} \cdot B^4 = 2C^4 + D^4$. Hence $(3A^2)^2 - D^4 = 2C^4 - 3 \cdot 2^{4k+2} \cdot B^4 \implies 2C^4 - 3 \cdot 2^{4k+2} \cdot B^4 \equiv 0 \bmod 8$ and this is absurd.

i2.) $=$ ii2.): $U_2 = \pm(3A^4 + 2^{4k+2} \cdot B^4) = \pm(\frac{C^4+2D^4}{3})$ and since the signs are not independent we have $(3A^2)^2 - C^4 = 2D^4 - 3 \cdot 2^{4k+2} \cdot B^4$. According to i2.) $=$ ii1.) this is impossible after congruence considerations.

i3.) $=$ ii1.): $U_2 = \pm(2^{4k+2} \cdot A^4 + 3B^4) = \pm(\frac{2C^4+D^4}{3})$ and since the signs are not independent we have

$$(3B^2)^2 - D^4 = 2C^4 - 3 \cdot 2^{4k+2} \cdot A^4$$

and again according to i2.) $=$ ii1.) this is impossible after congruence considerations.

i3.) $=$ ii2.): $U_2 = \pm(2^{4k+2} \cdot A^4 + 3B^4) = \pm(\frac{C^4+2D^4}{3})$ and since the signs are not independent we have $(3B^2)^2 - C^4 = 2D^4 - 3 \cdot 2^{4k+2} \cdot A^4$ and according to i2.) = ii1.) this is impossible after congruence considerations.

i4.) $=$ ii1.):

$$U_2 = \pm(3 \cdot 2^{4k+2} \cdot A^4 + B^4) = \pm(\frac{2C^4 + D^4}{3}) \tag{19}$$

and

$$U_3 = \pm(3 \cdot 2^{4k+2} \cdot A^4 - B^4) = \pm(2C^4 - D^4). \tag{20}$$

In analogy with the discussion performed in i1.) $=$ ii.1) we see that the signs in (19) and (20) are not independent. Hence from equation (20) we have $3 \cdot 2^{4k+2} \cdot A^4 - B^4 = 2C^4 - D^4 \implies 2C^4 - 3 \cdot 2^{4k+2} \cdot A^4 = D^4 - B^4$ and according to i1.) $=$ ii2.) this is impossible after congruence considerations.

i4.) $=$ ii2.):

$$U_2 = \pm(3 \cdot 2^{4k+2} \cdot A^4 + B^4) = \pm(\frac{C^4 + 2D^4}{3}) \tag{21}$$

and

$$U_3 = \pm(3 \cdot 2^{4k+2} \cdot A^4 - B^4) = \pm(C^4 - 2D^4). \tag{22}$$

In analogy with the discussion performed in i1.) $=$ ii1.) we see that the signs in (21) and (22) are not independent.

Hence equation (21) + (22) $\implies$

$$24 \cdot (2^k \cdot A)^4 = C^4 - B^4$$

which according to Lemma 1 has no non-zero solutions and again possible zero solutions violate the condition $x \cdot y \cdot z \neq 0$ in the title equation.

Thus we have shown that case II is impossible.

Case III. $3 \mid a$ and $3 \nmid b$

Hence $a = 3t$. From $y^2 = a(a^2 - 6b^2)$ we have $y^2 = 9t(3t^2 - 2b^2)$ and since the factors on the RHS are coprime we see that

$$t = \pm U_1^2 \tag{23}$$

and

$$3t^2 - 2b^2 = \pm U_3^2 \tag{24}$$

where U_1 and U_3 are odd since y is odd. Since $(b, U_3) = 1$ we reject the negative sign in (23) and (24) after reduction modulo 3 of equation (24). Hence

$$2b^2 = 3U_1^4 - U_3^2 \tag{25}$$

and we realize after congruence considerations that b must be odd. From $x^2 = b(3a^2 - 2b^2)$ it then follows since $(b, 3a^2 - 2b^2) = 1$ that

$$b = \pm U_4^2 \tag{26}$$

and

$$3a^2 - 2b^2 = \pm U_2^2. \tag{27}$$

Since $(b, U_2) = 1$ the negative sign in (26) and (27) is rejected after reduction modulo 3 of equation (27). From (23), (25), (26), (27) and since $a = 3t$ we have

$$3U_1^4 - 2U_4^4 = U_3^2 \tag{28}$$

and $3 \cdot (3U_1^2)^2 - 2U_4^4 = U_2^2 \Longrightarrow$

$$27U_1^4 - 2U_4^4 = U_2^2. \tag{29}$$

$(29) - (28) \Longrightarrow$

$$24U_1^4 = U_2^2 - U_3^2. \tag{30}$$

From (28) we have

$$27U_1^4 - 18U_4^4 = 9U_3^2. \tag{31}$$

$(29) - (31) \Longrightarrow$

$$16U_4^4 = U_2^2 - 9U_3^2 \tag{32}$$

From equation (30) we have,

$24U_1^4 = (U_2 + U_3) \cdot (U_2 - U_3)$. Thus according to case II since $(U_2, U_3) = 1$ we have the following possibilities,

i1.) $U_2 + U_3 = \pm 4A^4$ and $U_2 - U_3 = \pm 3 \cdot 2B^4$. Hence
$U_2 = \pm(2A^4 + 3B^4), U_3 = \pm(2A^4 - 3B^4)$ and $3 \nmid A$.

i2.) $U_2 + U_3 = \pm 3 \cdot 4A^4$ and $U_2 - U_3 = \pm 2B^4$. Hence
$U_2 = \pm(6A^4 + B^4), U_3 = \pm(6A^4 - B^4)$ and $3 \nmid B$.

i3.) $U_2 + U_3 = \pm 2A^4$ and $U_2 - U_3 = \pm 3 \cdot 4B^4$. Hence
$U_2 = \pm(A^4 + 6B^4), U_3 = \pm(A^4 - 6B^4)$ and $3 \nmid A$.

i4.) $U_2 + U_3 = \pm 3 \cdot 2A^4$ and $U_2 - U_3 = \pm 4B^4$. Hence
$U_2 = \pm(3A^4 + 2B^4), U_3 = \pm(3A^4 - 2B^4)$ and $3 \nmid B$.

$U_1 = A \cdot B$ where $(A, B) = 1$ since $(U_2, U_3) = 1$.

From equation (32) we have $16U_4^4 = (U_2 + 3U_3) \cdot (U_2 - 3U_3)$.Hence according to case II we have the following alternatives,

ii1.) $U_2 + 3U_3 = \pm 8C^4$ and $U_2 - 3U_3 = \pm 2D^4$. Hence
$U_2 = \pm(4C^4 + D^4)$ and $U_3 = \pm(\frac{4C^4 - D^4}{3})$.

ii2.) $U_2 + 3U_3 = \pm 2C^4$ and $U_2 - 3U_3 = \pm 8D^4$. Hence

$U_2 = \pm(C^4 + 4D^4)$ and $U_3 = \pm(\frac{C^4 - 4D^4}{3})$.

$U_4 = C \cdot D$ where $(C, D) = 1$ and $3 \nmid C \cdot D$ since $(U_2, 3U_3) = 1$.

N.B. Concerning the expressions of U_2 and U_3 in i1.) - i.4) and ii1.) - ii2.) we have $U_2 = \pm(.....)$ and $U_3 = \pm(.....)$. This certainly means that $U_2 = +(.....)$ and $U_3 = +(.....)$ or $U_2 = -(.....)$ and $U_3 = -(.....)$. As in case II we notice that if solutions exist at least one parametric solution of U_2 and U_3 in i1.) - i4.) must be equal to at least one parametric solution of U_2 and U_3 in ii1.) - ii2.) for some value (values) of A, B, C and D. Thus we have the following cases,

i1.) = ii1.):

$$U_2 = \pm(2A^4 + 3B^4) = \pm(4C^4 + D^4). \tag{33}$$

Since A^4, B^4, C^4 and D^4 are positive the signs in (33) are not independent.

$$U_3 = \pm(2A^4 - 3B^4) = \pm(\frac{4C^4 - D^4}{3}) \tag{34}$$

and since the signs in (33) are not independent so are the signs in (34) and we get $6A^4 - 9B^4 = 4C^4 - D^4 \Longrightarrow 6A^4 - 4C^4 = (3B^2)^2 - D^4$. So $6A^4 - 4C^4 \equiv 0 \bmod 8$ and this is impossible.

i1.) = ii2.):

$$U_2 = \pm(2A^4 + 3B^4) = \pm(C^4 + 4D^4). \tag{35}$$

Since A^4, B^4, C^4 and D^4 are positive the signs in (35) are not independent and we have

$$2A^4 + 3B^4 = C^4 + 4D^4. \tag{36}$$

$$U_3 = \pm(2A^4 - 3B^4) = \pm(\frac{C^4 - 4D^4}{3}). \tag{37}$$

and since the signs in (35) are not independent so are the signs in (37) and we get

$$6A^4 - 9B^4 = C^4 - 4D^4. \tag{38}$$

$(36) + (38) \Longrightarrow$

$$8A^4 - 6B^4 = 2C^4 \Longrightarrow 4A^4 - 3B^4 = C^4 \tag{39}$$

and since $(A, B) = 1$ we see that A, B and C are pairwise relatively prime. Hence according to an old theorem of E. Lucas [12] the only non-zero solutions to equation (39) are $A = \pm 1, B = \pm 1$ and $C = \pm 1$ and if these values are inserted in (36) or (38) we have $D = \pm 1$. Hence $U_2 = 5$ and $U_3 = -1$ or $U_2 = -5$ and $U_3 = 1$. Moreover from equation (32) we have $U_4^4 = 1 \Longrightarrow U_4 = \pm 1$. From equation (26) we get, after excluding the negative sign, $b = (\pm 1)^2 = 1$. Furthermore from equation (30) we see that $U_1 = \pm 1$ and from equation (23) we have after excluding the negative sign $t = (\pm 1)^2 = 1$. With $a = 3t$ we get $a = 3$. Finally if these values of a and b are inserted in the expressions of x^2 and y^2 previously we have $x^2 = 25 \Longrightarrow x = \pm 5$ and $y^2 = 9 \Longrightarrow y = \pm 3$ and since $z = a^2 + 2b^2$ we see that $z = 11$.

i2.) = ii1.): $U_2 = \pm(6A^4 + B^4) = \pm(4C^4 + D^4)$. Since A^4, B^4, C^4 and D^4 are positive we have $6A^4 + B^4 = 4C^4 + D^4$. Hence $D^4 - B^4 = 6A^4 - 4C^4$ and since A, B, C and D are all odd we have $6A^4 - 4C^4 \equiv 0 \bmod 16$ and this is impossible.

i2.) = ii2.): $U_2 = \pm(6A^4 + B^4) = \pm(C^4 + 4D^4)$ which according to i2.) = ii1.) must be impossible after congruence considerations.

i3.) $=$ ii1.): $U_2 = \pm(A^4 + 6B^4) = \pm(4C^4 + D^4)$ which according to i2.) $=$ ii1.) must be impossible after congruence considerations.

i3.) $=$ ii2.): $U_2 = \pm(A^4 + 6B^4) = \pm(C^4 + 4D^4)$ which according to i2.) $=$ ii1.) must be impossible after congruence considerations.

i4.) = ii1.):

$$U_2 = \pm(3A^4 + 2B^4) = \pm(4C^4 + D^4). \tag{40}$$

Since A^4, B^4, C^4 and D^4 are positive the signs in equation (40) are not independent and we have

$$3A^4 + 2B^4 = 4C^4 + D^4. \tag{41}$$

$$U_3 = \pm(3A^4 - 2B^4) = \pm(\frac{4C^4 - D^4}{3}). \tag{42}$$

and since the signs in (40) are not independent so are the signs in (42) and we get

$$9A^4 - 6B^4 = 4C^4 - D^4. \tag{43}$$

Equation $(41) - (43) \Longrightarrow 4B^4 - 3A^4 = D^4$ and in compliance with the discussion performed in i1.) = ii2.) this will ultimately lead to the only non-zero solutions $A = \pm 1$, $B = \pm 1$, $C = \pm 1$ and $D = \pm 1$. Hence $U_2 = 5$ and $U_3 = 1$ or $U_2 = -5$ and $U_3 = -1$. Thus according to i1.) $=$ ii2.) we have again $(x, y, z) = (\pm 5, \pm 3, 11)$.

i.4) = ii.2):

$$U_2 = \pm(3A^4 + 2B^4) = \pm(C^4 + 4D^4). \tag{44}$$

Since A^4, B^4, C^4 and D^4 are positive the signs in (22) are not independent.

$$U_3 = \pm(3A^4 - 2B^4) = \pm(\frac{C^4 - 4D^4}{3}). \tag{45}$$

and since the signs in (44) are not independent so are the signs in (45). Hence $(3A^2)^2 - C^4 = 6B^4 - 4D^4$ and according to i1.) = ii1.) this is impossible.

Thus there are only primitive non-zero solutions to the title equation in case III when i1.) $=$ ii2.) and i4.) $=$ ii1.) with $A = \pm 1, B = \pm 1, C = \pm 1$ and $D = \pm 1$ and this corresponds to the only primitive non-zero solutions to the title equation $2x^4 + y^4 = z^3$ namely $(x, y, z) = (\pm 5, \pm 3, 11)$. $\square$

We can now summarize the results in this work together with extensive results by others in the following theorem.

Theorem 2. *The only primitive non-zero solutions to the Diophantine equation* $2x^4 + y^4 = z^n$ *for all* $n \geqq 2$ *are* $(x, y, z, n) = (\pm 5, \pm 3, 11, 3)$.

Proof. The proof of no non-zero solutions if $n = 2$ can be found in e.g. [14]. If $n > 3$, we conclude from [4,5] that the only primitive positive non-zero solutions to the equation $2x^2 + y^4 = z^n$ for all $n > 3$ is $(x, y, z, n) = (11, 1, 3, 5)$ so the equation $2(x^2)^2 + y^4 = z^n$ has no non-zero solutions if $n > 3$ since 11 is not a square. $\square$

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