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On solutions of the Diophantine equation $\frac{1}{x_1}+\frac{2}{x_2}+\frac{3}{x_3}+\ldots+\frac{k}{x_k}=1$ when $2 \le x_1 < x_2 < x_3 < \ldots < x_k$ are integers and $k = x_1$

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Abstract: Some general solutions of the title equation are established and exhibited. It is also shown that for each value of x_1 when $x_1 \ge 4$, the equation has at least three solutions. For the particular values $x_1 = 2, 3, 4$, all the solutions of the equation are determined and demonstrated.
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1 Introduction

This article is concerned in finding solutions of the Diophantine equation

$$\begin{gathered} \frac{1}{x_1}+\frac{2}{x_2}+\frac{3}{x_3}+\cdots+\frac{k}{x_k}=1 \\ 2 \le x_1 < x_2 < x_3 < \ldots < x_k \\ k = x_1. \end{gathered} \tag{1}$$

Authors like B. M. Stewart, W. A. Webb [1, 2] and others have considered related problems and topics.

In Section 2, some general solutions of (1) are established and exhibited. In Section 3, all the solutions of (1) are determined and demonstrated for each of the values $x_1 = 2, 3, 4$. Moreover, it is shown that (1) has at least three solutions for each value of x_1 when $x_1 \ge 4$.

2 Some general solutions

We shall now establish and exhibit some general solutions of (1).

For every value $x_1 \geq 2$, the sum of x_1 unit fractions each of which equals to $\frac{1}{x_1}$ yields

$$\frac{1}{x_1}+\frac{1}{x_1}+\frac{1}{x_1}+\cdots+\frac{1}{x_1}=x_1\cdot\frac{1}{x_1}=1,$$

implying the equality

$$\frac{1}{x_1}+\frac{2}{2}\cdot\frac{1}{x_1}+\frac{3}{3}\cdot\frac{1}{x_1}+\cdots+\frac{x_1}{x_1}\cdot\frac{1}{x_1}=1,$$

or

$$\frac{1}{x_1}+\frac{2}{2x_1}+\frac{3}{3x_1}+\cdots+\frac{x_1}{x_1\cdot x_1}=1 \tag{2}$$

when $x_1 = x_1$, $x_2 = 2x_1$, $x_3 = 3x_1$, …, $x_k = x_1 . x_1$ are all distinct. Solution (2) is indeed the simplest solution of (1).

Hereafter, we denote (2) as the **Trivial Solution** of (1) for every value of x_1 when $x_1 \geq 2$. The identity

$$\frac{1}{N}=\frac{1}{N+M}+\frac{M}{N(N+M)}, \tag{3}$$

where $N \geq 1$, $M \geq 1$ are integers will be utilized in the following discussion.

We will now show that for any fixed value of x_1 when $x_1 \geq 2$, the **Trivial Solution** (2) immediately yields a solution of (1) with the smallest value being $x_1 + 1 < 2x_1$.

In (3), the smallest value of M is $M = 1$. Thus, the values $M = 1$ and $N = x_1$ yield

$$\frac{1}{x_1}=\frac{1}{x_1+1}+\frac{1}{x_1(x_1+1)} \tag{4}$$

Substituting (4) into the **Trivial Solution** (2) results in the equality.

$$\frac{1}{x_1+1}+\left[\frac{2}{2x_1}+\frac{3}{3x_1}+\cdots+\frac{x_1}{x_1\cdot x_1}\right]+\frac{x_1+1}{x_1+1}\cdot\frac{1}{x_1(x_1+1)}=1$$

or

$$\frac{1}{x_1+1}+\frac{2}{2x_1}+\frac{3}{3x_1}+\cdots+\frac{x_1}{x_1\cdot x_1}+\frac{x_1+1}{x_1(x_1+1)^2}=1 \tag{5}$$

a solution of (1) satisfying all the conditions.

Additional solutions are now established for values larger than $x_1 + 1$. In the **Trivial Solution** (2) substitute $\frac{1}{x_1 + T}$ for $\frac{1}{x_1}$, where T is an integer and $T = 2, 3, \ldots, x_1 - 1$. From (3), it follows that $\frac{1}{x_1} = \frac{1}{x_1 + T} + \frac{T}{x_1(x_1 + T)}$, and hence (2) results in

$$\frac{1}{x_1 + T} + \left[\frac{2}{2x_1} + \frac{3}{3x_1} + \cdots + \frac{x_1}{x_1 \cdot x_1}\right] + \frac{T}{x_1(x_1 + T)} = 1$$

or

$$\frac{1}{x_1 + T} + \left[\frac{2}{2x_1} + \cdots + \frac{x_1}{x_1 \cdot x_1}\right] + \left[\frac{x_1 + 1}{(x_1 + 1)x_1(x_1 + T)} + \cdots + \frac{x_1 + T}{(x_1 + T)x_1(x_1 + T)}\right] = 1 \qquad (6)$$

a solution of (1).

Observe that (6) is an identity which holds for any fixed value of x_1, and each value of T. It follows therefore that it represents a set of solutions for various values of $\frac{1}{x_1 + T}$.

3 All the solutions of equation (1) when $x_1 = 2, 3, 4$

In this section, we shall determine and exhibit all the solutions of (1) when $x_1 = 2$, $x_1 = 3$ and $x_1 = 4$. This is demonstrated in the following Theorem 1.

Theorem 1. *The Diophantine equation in positive integers*

$$\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} + \cdots + \frac{k}{x_k} = 1 \quad 2 \le x_1 < x_2 < x_3 < \cdots < x_k \quad k = x_1$$

has

(i) *one solution if* $x_1 = 2$,

(ii) *exactly two solutions if* $x_1 = 3$,

(iii) *exactly ten solutions if* $x_1 = 4$.

Proof. Suppose (i), i.e., $x_1 = 2$. Then we have $\frac{1}{2} + \frac{2}{x_2} = 1$ and $x_2 = 4$. Thus,

Solution 1. $$\frac{1}{2} + \frac{2}{4} = 1,$$

which is the **Trivial Solution** for $x_1 = 2$.

Suppose (ii), i.e. $x_1 = 3$. Then from (1), $\frac{1}{3} + \frac{2}{x_2} + \frac{3}{x_3} = 1$.

One can easily verify that the above equation has no solutions when $x_2 = 5$ and also when $x_2 > 6$. Hence, the only possible two values of x_2 are then $x_2 = 4$ and $x_2 = 6$. The respective two solutions are:

Solution 2. $$\frac{1}{3}+\frac{2}{4}+\frac{3}{18}=1,$$

and

Solution 3. $$\frac{1}{3}+\frac{2}{6}+\frac{3}{9}=1,$$

as the **Trivial Solution** for $x_1 = 3$.

The above three solutions verify parts (i) and (ii).

Suppose (iii), i.e. $x_1 = 4$. We have from (1)

$$\frac{1}{4}+\frac{2}{x_2}+\frac{3}{x_3}+\frac{4}{x_4}=1, \quad x_2 < x_3 < x_4. \tag{7}$$

First, when $x_2 \geq 11$, it is shown that (7) has no solutions. Secondly, the same is also true for $x_2 = 10$ and $x_2 = 9$, but for different reasons.

Since $\frac{3}{4} > \frac{2}{11}+\frac{3}{12}+\frac{4}{13}$, it therefore follows that $x_2 < 11$.

If $x_2 = 10$, (7) yields $\frac{11}{20}=\frac{3}{x_3}+\frac{4}{x_4}$. Since $\frac{11}{20} > \frac{3}{13}+\frac{4}{14}$, therefore $x_3 = 11$ and $x_3 = 12$ are the only possible values. It is now easily seen that when $x_3 = 11, 12$ no integer x_4 exists. Thus $x_2 \neq 10$.

If $x_2 = 9$, (7) implies that $\frac{19}{36}=\frac{3}{x_3}+\frac{4}{x_4}$. Since $\frac{19}{36} > \frac{3}{13}+\frac{4}{14}$, it follows that $10 \leq x_3 \leq 12$. But, when $x_3 = 10, 11, 12$, one can easily verify that x_4 is not an integer. Hence $x_2 \neq 9$.

Therefore, in (7) x_2 may assume any of the values $x_2 = 5, 6, 7, 8$. Each of these values will now be considered separately.

Let $x_2 = 5$. From (7) it follows that $\frac{7}{20}=\frac{3}{x_3}+\frac{4}{x_4}$. This equality is false when $x_3 \geq 20$, and therefore $6 \leq x_3 \leq 19$. If $x_3 = 6, 7, 8$, then $\frac{3}{x_3} > \frac{7}{20}$, and hence $9 \leq x_3 \leq 19$. For $x_3 = 9$, $x_3 = 10$ and $x_3 = 12$, the three solutions of (1) are:

Solution 4. $$\frac{1}{4}+\frac{2}{5}+\frac{3}{9}+\frac{4}{240}=1,$$

Solution 5. $$\frac{1}{4}+\frac{2}{5}+\frac{3}{10}+\frac{4}{80}=1,$$

Solution 6. $$\frac{1}{4}+\frac{2}{5}+\frac{3}{12}+\frac{4}{40}=1.$$

For each of the following values $x_3 = 11$ and $13 \leq x_3 \leq 19$, one can easily verify that x_4 is not an integer.

The case $x_2 = 5$ is complete.

Let $x_2 = 6$. From (7) we have $\frac{5}{12} = \frac{3}{x_3} + \frac{4}{x_4}$. The equality is false when $x_3 \geq 17$. Hence, $7 \leq x_3 \leq 16$. Since $\frac{3}{7} > \frac{5}{12}$, therefore $x_3 \neq 7$, and $8 \leq x_3 \leq 16$.

The values $x_3 = 8$, $x_3 = 9$ and $x_3 = 12$ yield three solutions of (1):

Solution 7. $$\frac{1}{4} + \frac{2}{6} + \frac{3}{8} + \frac{4}{96} = 1,$$

Solution 8. $$\frac{1}{4} + \frac{2}{6} + \frac{3}{9} + \frac{4}{48} = 1,$$

Solution 9. $$\frac{1}{4} + \frac{2}{6} + \frac{3}{12} + \frac{4}{24} = 1.$$

For each of the following values $x_3 = 10$, 11 and $13 \leq x_3 \leq 16$, it is easily seen that x_4 is not an integer.

This concludes the case $x_2 = 6$.

Let $x_2 = 7$. Then (7) yields $\frac{13}{28} = \frac{3}{x_3} + \frac{4}{x_4}$. The equality is false when $x_3 \geq 15$. Therefore, $8 \leq x_3 \leq 14$. For $x_3 = 14$, the solution of (1) is

Solution 10. $$\frac{1}{4} + \frac{2}{7} + \frac{3}{14} + \frac{4}{16} = 1.$$

For each of the values $8 \leq x_3 \leq 13$, it follows that x_4 is not an integer.

The case $x_2 = 7$ is complete.

Let $x_2 = 8$. From (7) we have $\frac{1}{2} = \frac{3}{x_3} + \frac{4}{x_4}$. This equality is false when $x_3 \geq 14$. Thus, $9 \leq x_3 \leq 13$. The values $x_3 = 9$, $x_3 = 10$ and $x_3 = 12$ imply the following three solutions of (1), namely:

Solution 11. $$\frac{1}{4} + \frac{2}{8} + \frac{3}{9} + \frac{4}{24} = 1,$$

Solution 12. $$\frac{1}{4} + \frac{2}{8} + \frac{3}{10} + \frac{4}{20} = 1,$$

Solution 13. $$\frac{1}{4} + \frac{2}{8} + \frac{3}{12} + \frac{4}{16} = 1,$$

which is also the **Trivial Solution**. If $x_3 = 11$ and $x_3 = 13$, then x_4 is not an integer.

This concludes the case $x_2 = 8$.

Solutions 4 – 13 establish part (iii).

The proof of Theorem 1 is complete. □

Finally, we show that (1) has at least three solutions for each value of x_1 when $x_1 \geq 4$. This is demonstrated in the following three solutions, namely **Solution A**, **Solution B** and **Solution C** the **Trivial Solution**.

For $x_1 = 4$, (1) has at least three solutions. Therefore, let $x_1 \geq 5$ be any fixed value. The two slightly modified solutions (6) and (5), together with solution (2), respectively yield the above mentioned three solutions.

Solution A. $$\frac{1}{x_1} + \frac{2}{2(x_1-2)} + \cdots + \frac{x_1-2}{(x_1-2)(x_1-2)} + \frac{x_1-1}{(x_1-1)(x_1-2)x_1} + \frac{x_1}{x_1(x_1-2)x_1} = 1,$$

Solution B. $$\frac{1}{x_1} + \frac{2}{2(x_1-1)} + \cdots + \frac{x_1-1}{(x_1-1)(x_1-1)} + \frac{x_1}{x_1(x_1-1)x_1} = 1,$$

Solution C. $$\frac{1}{x_1} + \frac{2}{2x_1} + \cdots + \frac{x_1}{x_1 \cdot x_1} = 1.$$

Evidently, more solutions of (1) exist for each value of x_1.

References

[1] Stewart, B. M., & Webb, W. A. (1966) Sums of fractions with bounded numerators, *Canadian J. Math*., 18, 999–1003.

[2] Webb, W. A. (1976) On the Diophantine equation $\frac{k}{n} = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3}$, *Časopis pro Pěstování Matematiky*, 101, 360–365.