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Study of some equivalence classes of primes

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Abstract: In this paper, we introduce an equivalence relation $\sim$ on $\mathbb{P}$ for studying some specific classes of prime numbers. This relation and the famous prime number theorem allows us to estimate the number of prime numbers of each equivalence class, the number of the different equivalence classes and to show some other results.

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1 Introduction

Let $\mathbb{P}$ be the set of prime numbers, and for all $x \in \mathbb{R}$, let $\pi(x)$ be the prime-counting function. The prime number theorem which was proved independently by de la Vallée Poussin [1], and Hadamard [2] in 1896, states that:

$$\pi(x) \sim \frac{x}{\ln x}, \text{ as } x \to +\infty. \tag{1}$$

We can give an equivalent statement for this theorem as, for example, let p_n denote the n-th prime number. Then

$$\pi^{-1}(n) = p_n \sim n \ln n \; as \; n \to +\infty. \tag{2}$$

We define the following functions:

$$f^n(x) = \overbrace{f \circ f \circ \ldots \circ f(x)}^{n \text{ times}} \text{ and } f^{-n}(y) = \overbrace{f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}(y)}^{n \text{ times}},$$

with $f^0(x) = x$ and o is the composition operator.

The aim of this paper is to construct an equivalence relation $\sim$ on the set of primes, using the restriction of the prime-counting function π to $\mathbb{P}$, for studying some classes of primes. The following results illustrate asymptotic distribution of a number of functions that we have proposed.

2 Main results

We start with the following obvious lemma:

Lemma 2.1. *Let $\widetilde{\pi}$ be a restriction of π to $\mathbb{P}$. Then, $\widetilde{\pi} : \mathbb{P} \to \mathbb{N}$ is a bijection and its inverse is $\widetilde{\pi}^{-1} : \mathbb{N} \to \mathbb{P}$.*

Remark. Throughout this paper, we simply use the notation π to designate the restriction of π to the set $\mathbb{P}$ instead of using $\widetilde{\pi}$.

The proof of the following theorem is obvious.

Theorem 2.1.

1. *We define the relation $\sim$ on the set of prime numbers $\mathbb{P}$ defined by: if p and q are two prime numbers, $p \sim q$ if and only if there exists $n \in \mathbb{Z}$ such that $p = \pi^n(q)$. Then, $\sim$ is an equivalence relation. The elements of the equivalence class $\dot{p}$ are defined by:*

$$\dot{p} = \{\ldots, \pi^2(p), \pi(p), p, \pi^{-1}(p), \pi^{-2}(p), \ldots\}.$$

2. *The smallest element p_0 of $\dot{p}$ is the prime number which verify $\pi(p_0)$ is not prime and it is called origin of the class $\dot{p}$.*

Example 1.

- $\dot{31} = \{2, 3, 5, 11, 31, 127, 709, 5381, 52711, \ldots\}$ with origin the number 2 since $\pi(2) = 1$ is not a prime.
- $\dot{7} = \{7, 17, 59, 277, 1787, \ldots\}$ with origin the number 7 since $\pi(7) = 4$ is not a prime.

Notation. We denote by $\mathcal{P}_0(x)$ the set of all origins $p_0 \leq x$ defined by:

$$\mathcal{P}_0(x) = \{2, 7, 13, 19, 23, 29, 37, 43, 47, \ldots\}.$$

And we denote simply by $\mathcal{P}_0$ the set of all origins p_0.

Theorem 2.2. *Let A be a finite set of increasing sequence of consecutive primes defined by:*

$$A = \{p_i, 1 \leq i \leq n \textit{ and } p_1 = 5\}.$$

Then, there exists at least one class $\dot{p} \subset A, p \in A$, such that $\dot{p} = \{p\}$.

Before giving the proof of this result, we give an illustrative example.

Example 2. Let A be a set which defined by

$$A = \{5, 7, 11, 13, 17\}.$$

- The class of the integer 5 is $\dot{5} = \{5, 11\}$ and its cardinality is greater than 1.

- We notice that $7 \notin \dot{5}$, therefore the integer 7 constitutes the origin of a new class which is $\dot{7} = \{7, 17\}$ and its cardinality is greater than 1. It only remains to see that the prime number 13 does not belong to the classes $\dot{2}$ and $\dot{7}$. Thus the prime number 13 constitutes the origin of a new class $\dot{13}$ and clearly its cardinality is 1.

Proof of theorem 2.2. Assuming that each class $\dot{p_i}, 1 \leq i \leq n$, containing at least two elements, and supposing that $\pi(p_n)$ is prime. According to the prime number theorem, the interval $[\pi(p_n), p_n]$ contains $\pi(p_n) - \pi^2(p_n) = k$ prime numbers. This leads to the two following cases:

- If $k = 1$, i.e., there exists only one prime number in $[\pi(p_n), p_n]$, namely q. Therefore, we obtain $\pi^{-1}(q) \notin A$ and $\pi(q)$ is not a prime number since $\pi(q)$ and $\pi(p_n)$ are consecutive. Then $\dot{q}$ contains only one element in A which is the prime number q.

- If $k > 1$, i.e., there exist at least two prime numbers in $[\pi(p_n), p_n]$, namely $q_1, \ldots, q_k$. We suppose that $q_t, t \in \{1, \ldots, k\}$, is a prime number. Then q_{t+1} is not prime and since $\pi^{-1}(q_{t+1}) \notin A$, then, the unique element of $\dot{q_{t+1}}$ is q_{t+1}.

Finally, in both cases, there exist at least one class $\dot{p}, p \in A$, such that p is a unique element of this class in A, i.e., $\dot{p} = \{p\}$. The proof now is completed. □

We have the following definition.

Definition 1. Let A be the set defined as in the Theorem 2.2 and p is a prime number belonging to A. We say that $\dot{p}$ is an *outside* class of A, if $\pi(p)$ is not prime in A and $\pi^{-1}(p) \notin A$ i.e., $|\dot{p}| = 1$ in A. In the case where $|\dot{p}| \geq 2$, we say that $\dot{p}$ is an *inside* class of A.

Lemma 2.2. *1. Let (x_n) be a sequence defined by the recursive relation:*

$$\begin{cases} x_{n+1} = p(x_n) = \frac{x_n}{\ln x_n}, \\ x_0 \geq e \approx 2.7182818. \end{cases}$$

Then

$$p(x_0) = e \prod_{i=0}^{\infty} \ln p(x_i) = e \prod_{i=0}^{\infty} \ln p^{i+1}(x_0), \tag{3}$$

Proof. 1. We set $\ln \ln x = \ln_2 x$ and we have

$$x_1 = p(x_0) = \frac{x_0}{\ln x_0} \Longrightarrow \ln x_1 = \ln x_0 - \ln_2 x_0$$
$$x_2 = p(x_1) = \frac{x_1}{\ln x_1} \Longrightarrow \ln x_2 = \ln x_1 - \ln_2 x_1$$
$$\vdots$$
$$x_{n+1} = p(x_n) = \frac{x_n}{\ln x_n} \Longrightarrow \ln x_{n+1} = \ln x_n - \ln_2 x_n.$$

And combining all these, we obtain

$$\ln x_{n+1} = \ln p(x_n) = \ln x_0 - \sum_{i=0}^{n} \ln_2 x_i,$$

which is equivalent to

$$\frac{x_0}{x_{n+1}} = \prod_{i=0}^{n} \ln x_i.$$

Then, passing to the limit, we obtain

$$\lim_{n \longrightarrow +\infty} \frac{x_0}{x_{n+1}} = \lim_{n \to \infty} \frac{x_0}{p(x_n)} = \lim_{n \longrightarrow +\infty} \prod_{i=0}^{n} \ln x_i = \lim_{n \to \infty} \ln x_0 \prod_{i=0}^{n-1} \ln p(x_i).$$

Consequently,

$$\frac{x_0}{e} = \prod_{i=0}^{\infty} \ln x_i = \ln x_0 \prod_{i=0}^{\infty} \ln p(x_i), \tag{4}$$

and since

$$\lim_{n \longrightarrow +\infty} x_n = e.$$

The formula (4) is obviously equivalent to

$$p(x_0) = e \prod_{i=0}^{\infty} \ln p(x_i),$$

which is the desired result.

□

Lemma 2.3. *Let $x > 0$. We define the number of prime numbers belonging to the class $\dot{p_0}$ less than or equal to x as follows*

$$\pi_{iter}(x, p_0) = \sum_{\substack{p_0 \le p \le x \\ p \in \dot{p_0}}} 1 = \sum_{\substack{p_0 \le p \le x \\ p = \pi^{-l}(p_0)}} 1, l \in \mathbb{N}.$$

Then, $\pi_{iter}(x, p_0)$ is approximately equal to the number of iterations n of the sequence $x_{n+1} = p(x_n)$ with $x_0 = x$ and the value of n verify $x_n \ge p_0$ and $x_{n+1} < p_0$, as $x \to +\infty$.

Proof. From the prime number theorem, we have

$$\pi(x) \sim p(x) = \frac{x}{\ln(x)}, x \to \infty \Rightarrow \pi^n(x) \sim p^n(x), x \to \infty.$$

Next, we can choose an integer n such that $\pi^n(x) = p_0$ with $\pi(p_0)$ not a prime number, it follows that

$$p_0 \sim p^n(x), x \to \infty.$$

And,

$$\pi_{iter}(x, p_0) = \sum_{\substack{p_0 \le p \le x \\ p \in \dot{p_0}}} 1 = \sum_{\substack{p_0 \le p \le x \\ p = \pi^{-l}(p_0)}} 1 \sim \sum_{\substack{p_0 \le p \le x \\ p \approx p^{-l}(p_0)}} 1 = \sum_{l=1}^{n(x,p_0)} 1 = n(x, p_0) \text{ as } x \to \infty,$$

such that $n(x, p_0)$ is the number of iterations n which depends obviously on x and p_0. □

Theorem 2.3. *Let $x > 0$. We define the function $\eta(x, p_0)$ as follows:*

$$\eta(x,p_0) = \sum_{\substack{p_0 \le p \le x,\\ p \in \dot{p_0}}} \ln\ln p.$$

Then, we have

$$\eta(x,p_0) \sim \ln x + o(\ln x).$$

Proof. In view of the proof of Lemma 2.3, we have,

$$p_0 = \pi^n(x) \sim p^n(x) \text{ implies that, } \ln x_0 \prod_{i=0}^{n} \ln \pi(x_i) \sim \ln x_0 \prod_{i=0}^{n} \ln p(x_i) \sim \frac{x_0}{p_0} = \frac{x}{p_0},$$

which is equivalent to

$$\ln\ln x_0 + \sum_{i=0}^{n} \ln\ln \pi(x_i) = \ln\ln x + \sum_{i=0}^{n} \ln\ln \pi(x_i) = \sum_{\substack{p_0 \le p \le x,\\ p \in \dot{p_0}}} \ln\ln p \sim \ln x - \ln p_0.$$

Finally, for all fixed p_0 and $x \longrightarrow \infty$, we have $-\ln p_0 / \ln x \longrightarrow 0$, then

$$\sum_{\substack{p_0 \le p \le x,\\ p \in \dot{p_0}}} \ln\ln p \sim \ln x + o(\ln x).$$

i.e., $\eta(x,p_0) \sim \ln x + o(\ln x)$. □

Theorem 2.4. *We have,*

1.
$$\pi_{iter}(x,p_0) \sim \frac{\eta(x,p_0)}{\ln\ln x}, x \to \infty.$$

2.
$$\pi_{iter}(x,p_0) \sim \pi_{iter}(x) \sim \frac{\ln x}{\ln\ln x} + o\left(\frac{\ln x}{\ln\ln x}\right), x \longrightarrow \infty.$$

Proof. 1. For the proof of the first formula, we have, on the one hand

$$\eta(x,p_0) = \sum_{\substack{p_0 \le p \le x,\\ p \in \dot{p_0}}} \ln\ln p \le \ln\ln x \sum_{\substack{p_0 \le p \le x,\\ p \in \dot{p_0}}} 1 = \pi_{iter}(x,p_0) \ln\ln x.$$

Then we obtain

$$\pi_{iter}(x,p_0) \ge \frac{\eta(x,p_0)}{\ln\ln x}.$$

On the other hand, for all $x > e, x^\delta > p_0$ with $0 < \delta < 1$, we have

$$\begin{aligned}
\eta(x,p_0) &\ge \ln\ln x^\delta \sum_{\substack{x^\delta < p \le x,\\ p \in \dot{p_0}}} 1 \\
&= (\ln\delta + \ln\ln x) \sum_{\substack{x^\delta < p \le x,\\ p \in \dot{p_0}}} 1 \\
&= (\ln\delta + \ln\ln x)(\pi_{iter}(x,p_0) - \pi_{iter}(x^\delta, p_0)) \\
&\ge (\ln\delta + \ln\ln x)(\pi_{iter}(x,p_0) - (\ln x)^\delta).
\end{aligned}$$

Then

$$\pi_{iter}(x, p_0) \leq (\ln x)^\delta + \frac{\eta(x, p_0)}{\ln \delta + \ln \ln x}.$$

Now, according to Lemma 2.3, $(\ln x)^\delta = o(\pi_{iter}(x, p_0))$, and then for x sufficiently large (depending on δ), $(\ln x)^\delta \leq (1-\delta)\pi_{iter}(x, p_0)$, and thus

$$\pi_{iter}(x, p_0) \leq \frac{\eta(x, p_0)}{\delta(\ln \delta + \ln \ln x)}.$$

Now, for all $\epsilon > 0$, we can choose δ more near to 1, for this δ, so that $1/\delta = 1 + \epsilon$, and for x sufficiently large, we have

$$\pi_{iter}(x, p_0) < (1+\epsilon)\frac{\eta(x, p_0)}{\ln \ln x}.$$

2. According to Theorem 2.3, we have

$$\pi_{iter}(x, p_0) \sim \frac{\eta(x, p_0)}{\ln \ln x} \sim \frac{\eta(x, p_0)}{\ln \ln x} \sim \frac{\ln x}{\ln \ln x} + o\left(\frac{\ln x}{\ln \ln x}\right) \sim \pi_{iter}(x), x \to \infty.$$

□

Definition 2. 1. Let x be a positive real number. We denote by $\pi_c(x)$ the number of *different* classes $\dot{p}$ such that $2 \leq p \leq x$. Precisely,

$$\pi_c(x) := \sum_{p_0 \leq x} 1, \ p_0 \in \mathcal{P}_0.$$

2. We denote by $\theta_0(x)$ the function defined by

$$\theta_0(x) = \sum_{p_0 \leq x} \ln p_0.$$

Example 3. In the interval $[2, 11]$, the value 2 represents the origin of the class $\dot{2}$ but the values $3, 5, 11$ do not, since they belong to the same class $\dot{2}$. The value 7 represent the origin of the class $\dot{7}$. Thus in this case, we have $\pi_c(x) = 2$.

We have the following result:

Theorem 2.5. *1. We have,*

$$\lim_{x\to\infty} \pi_c(x) = +\infty.$$

2. Let $p_0 \in [2, x]$. Then

$$\pi_c(x) = \pi(x) - \pi(\pi(x)) = \pi(x) - \pi^2(x). \tag{5}$$

Proof. 1. Suppose that the number of different classes is finite as $x \to \infty$. We know that

$$\sum_{i=0}^{\infty} \frac{1}{p_i} = \infty. \tag{6}$$

Next, let $p_i^k \in \dot{p_k}$, where k is finite by hypothesis. We obtain

$$\sum_{i=0}^{\infty} \frac{1}{p_i} = \sum_{k<\infty} \sum_{i=0}^{\infty} \frac{1}{p_i^k}.$$

Therefore, we have $\sum_{i=0}^{\infty} \frac{1}{p_i^k} < \infty$, and since the second sum has a finite number of terms, we deduce

$$\sum_{k<\infty} \sum_{i=0}^{\infty} \frac{1}{p_i^k} < \infty,$$

which contradicts formula (6).

2. To find the value of $\pi_c(x)$ means that we estimate the number of origins p_0. Clearly, the prime number p_0 is an origin that means $\pi(p_0)$ is not a prime number. Then, let p_0 be between 2 and x, and let $p_{0,x} \in \mathcal{P}_0$ be the greatest prime number in $[2, x]$, therefore $\pi(p_{0,x})$ is not a prime number and for all integer $l \geq 1$, $\pi^{-l}(\pi(p_{0,x}) + 1) > p_{0,x}$. So, we only have to search the numbers which are not primes and less than $\pi(p_{0,x})$. Thus, we have

   - The number of the even numbers less than or equal to $p_{0,x}$ equal to $\frac{\pi(p_{0,x})}{2}$.
   - the number of the odd numbers less than or equal to $p_{0,x}$ equal to $\frac{\pi(p_{0,x})}{2} - \pi(\pi(p_{0,x}))$ such that $\pi(\pi(p_{0,x}))$ is the number of the prime numbers less than or equal to $\pi(p_{0,x})$.

   Next, we add the two quantities, we obtain, since $p_{0,x} \leq x$, the quantity $\pi_c(x)$ which is equal to

$$\pi_c(x) = \pi(x) - \pi(\pi(x)).$$

□

We easily obtain the following consequence.

Corollary 2.1. *Let p be a prime and suppose that $p_0 = \pi^n(p)$ be an origin. Then*

$$p_0 = \pi(p) - \sum_{i=0}^{n-2} \pi_c\left(\pi^i(p)\right).$$

Proof. We have

$$\begin{aligned}
&\pi_c(x) = \pi(x) - \pi(\pi(x)) \\
&\pi_c(\pi(x)) = \pi(\pi(x)) - \pi(\pi^2(x)) \\
&\vdots \\
&\pi_c(\pi^{n-2}(x)) = \pi(\pi^{n-1}(x)) - \pi(\pi^n(x)).
\end{aligned}$$

Now, we add these equations together, we obtain the desired result. □

For all initial value $y_0 \gg e$, we define the following sequence:

$$y_{n+1} = y_n \ln y_n. \tag{7}$$

This sequence is stationary for $y_0 = e$ and increasing divergent to infinite for $y_0 > e$ and as $n \to \infty$. It is clear that, inductively, $y_n \geq y_0(\ln y_0 \times \ln y_0 \times \ldots \times \ln y_0) = y_0(\ln y_0)^n$, then we have the following consequence:

Lemma 2.4. *We have,*

$$\pi_c(x) \leq \frac{x \ln\ln x}{\ln^2 x} + \frac{\sum_{p_0 \leq x} \ln p_0}{\ln x} = \frac{x \ln\ln x}{\ln^2 x} + \frac{\theta_0(x)}{\ln x},$$

where p_0 represents the origin of the classes $\dot{p}$.

Proof. Since for all $p = p_n, n > 0$, we have $p > p_0 \ln^n p_0$, then

$$\sum_{p \leq x} \ln\ln p \geq \sum_{p_0 \ln^n p_0 \leq x, n} \ln\ln p_0 = \sum_{p_0, n \leq \frac{\ln x - \ln p_0}{\ln\ln p_0}} \ln\ln p_0 = \sum_{p_0 \leq x} \ln\ln p_0 \left\lfloor \frac{\ln x - \ln p_0}{\ln\ln p_0} \right\rfloor$$

$$\sim \sum_{p_0 \leq x} (\ln x - \ln p_0) = (\ln x)\pi_c(x) - \sum_{p_0 \leq x} \ln p_0$$

According to the following formula

$$\sum_{p \leq x} f(p) \approx \sum_{n \leq x} \frac{f(n)}{\ln n} \approx \int_2^x \frac{f(t)}{\ln t} dt. \tag{8}$$

we have

$$\sum_{p \leq x} \ln\ln p \approx \frac{x \ln\ln x}{\ln x},$$

therefore, the inequality is obtained directly by substitution. □

Proposition 2.1. *We have,*

1.

$$x\left(1 - \frac{1}{\ln x - \ln\ln x} - \frac{\ln\ln x}{\ln x}\right) \leq \theta_0(x) \leq x\left(1 - \frac{1}{\ln x - \ln\ln x}\right) \tag{9}$$

2.

$$\theta_0(x) \sim x \textit{ as } x \to \infty.$$

Proof. 1. We have

$$\pi(x) - \pi(\pi(x)) \leq \frac{x \ln\ln x}{\ln^2 x} + \frac{\sum_{p_0 \leq x} \ln p_0}{\ln x} \Rightarrow \sum_{p_0 \leq x} \ln p_0 \geq (\pi(x) - \pi(\pi(x))) \ln x - \frac{x \ln\ln x}{\ln x}.$$

Moreover, since

$$\sum_{p_0 \leq x} \ln p_0 \leq \ln x \sum_{p_0 \leq x} 1 = \pi_c(x) \ln x.$$

And recalling that

$$\pi(x) \sim \frac{x}{\ln x},$$

we obtain

$$\sum_{p_0 \leq x} \ln p_0 \geq \left(\frac{x}{\ln x} - \frac{x}{\ln x(\ln x - \ln\ln x)}\right) \ln x - \frac{x \ln\ln x}{\ln x} = x\left(1 - \frac{1}{\ln x - \ln\ln x} - \frac{\ln\ln x}{\ln x}\right).$$

The second inequality is obtained in the same way,

$$\sum_{p_0 \leq x} \ln p_0 \leq \ln x \left(\frac{x}{\ln x} - \frac{x}{\ln x(\ln x - \ln\ln x)}\right) = x\left(1 - \frac{1}{\ln x - \ln\ln x}\right).$$

2. It is enough to tend x to $+\infty$ in the inequality (9).

□

3 Future work

1. Our future work is to generalize this equivalence relation by introducing a new function

$$\phi \circ \pi : \mathbb{P} \to \mathbb{N},$$

   with ϕ a bijection defined in $\mathbb{N}$ to $\mathbb{N}$.

2. Study of the following hypothesis: let $p \in \mathbb{P}$. Suppose that the 2-tuple $(p, p+h)$ are primes infinitely often for all $h \in 2\mathbb{N}$. Then, there exists at least an integer $\overline{h}$ among these integers h verify the following equation: $\pi(p+\overline{h}) - \pi(p) = c$ infinitely often, such that $\pi(p+\overline{h})$ and $\pi(p)$ are primes and $c < \overline{h}$ fixed.

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