Conditions equivalent to the Descartes–Frenicle–Sorli Conjecture on odd perfect numbers

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Abstract: The Descartes-Frenicle-Sorli conjecture predicts that \( k = 1 \) if \( q^k n^2 \) is an odd perfect number with Euler prime \( q \). In this note, we present some conditions equivalent to this conjecture.

Keywords: Odd perfect number, Abundancy index, Deficiency.

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1 Introduction

If \( N \) is a positive integer, then we write \( \sigma(N) \) for the sum of the divisors of \( N \). A number \( N \) is perfect if \( \sigma(N) = 2N \). We denote the abundancy index \( I \) of the positive integer \( w \) as \( I(w) = \sigma(w)/w \). We also denote the deficiency \( D \) of the positive integer \( x \) as \( D(x) = 2x - \sigma(x) \) [11].

Euclid and Euler showed that that an even perfect number \( E \) must have the form

\[
E = (2^p - 1) 2^{p-1},
\]

where \( 2^p - 1 \) is a Mersenne prime. On the other hand, Euler showed that an odd perfect number \( O \) must have the form

\[
O = q^k n^2,
\]
where \( q \) is an Euler prime (i.e., \( q \equiv k \equiv 1 \pmod{4} \) and \( \gcd(q, n) = 1 \)).

It is currently unknown whether there are any odd perfect numbers. On the other hand, only 49 even perfect numbers have been found, a couple of which were discovered by the Great Internet Mersenne Prime Search [9]. It is conjectured that there are infinitely many even perfect numbers, and that there are no odd perfect numbers.

Descartes, Frenicle and subsequently Sorli conjectured that \( k = 1 \) [1]. Sorli conjectured \( k = 1 \) after testing large numbers with eight distinct prime factors for perfection [14].

Holdener presented some conditions equivalent to the existence of odd perfect numbers in [10]. In this paper, we prove the following results:

**Lemma 1.1.** If \( N = q^k n^2 \) is an odd perfect number with Euler prime \( q \), then \( k = 1 \) if and only if
\[
\frac{\sigma(n^2)}{q} \mid n^2.
\]

**Lemma 1.2.** If \( N = q^k n^2 \) is an odd perfect number with Euler prime \( q \), then
\[
I(n^2) \leq 2 - \frac{5}{3q}.
\]

**Lemma 1.3.** If \( N = q^k n^2 \) is an odd perfect number with Euler prime \( q \), then \( k = 1 \) if and only if
\[
D(n^2) \mid n^2.
\]

**Theorem 1.1.** If \( N = q^k n^2 \) is an odd perfect number with Euler prime \( q \), then
\[
I(n^2) = 2 - \frac{5}{3q}
\]
holds if and only if \( k = 1 \) and \( q = 5 \).

All of the proofs given in this note are elementary.

## 2 Preliminaries

Let \( N = q^k n^2 \) be an odd perfect number with Euler prime \( q \).

First, we show that the following equations hold.

**Lemma 2.1.** If \( N = q^k n^2 \) is an odd perfect number with Euler prime \( q \), then
\[
\gcd\left(n^2, \sigma(n^2)\right) = \frac{D(n^2)}{\sigma(q^{k-1})} = \frac{\sigma(N/q^k)}{q^k}.
\]

**Proof.** Since \( N = q^k n^2 \) is an odd perfect number, we have
\[
\sigma(q^k)\sigma(n^2) = \sigma(N) = 2N = 2q^k n^2,
\]
from which it follows that \( q^k \mid \sigma(n^2) \) (because \( \gcd\left(q^k, \sigma(q^k)\right) = 1 \)). Hence,
\[
\frac{\sigma(n^2)}{q^k} = \frac{\sigma(N/q^k)}{q^k}.
\]
is an integer.

First, we prove that

\[ \frac{D(n^2)}{\sigma(q^{k-1})} = \frac{\sigma(N/q^k)}{q^k}. \]

We rewrite the equation

\[ \sigma(q^k)\sigma(n^2) = 2^k n^2 \]

as

\[
\begin{align*}
(q^k + \sigma(q^{k-1}))\sigma(n^2) &= 2^k n^2 \\
\sigma(q^{k-1})\sigma(n^2) &= q^k \left(2n^2 - \sigma(n^2)\right) = q^k \cdot D(n^2) \\
\frac{\sigma(n^2)}{q^k} &= \frac{D(n^2)}{\sigma(q^{k-1})},
\end{align*}
\]

and we are done.

Next, we show that

\[ \gcd(n^2, \sigma(n^2)) = \frac{D(n^2)}{\sigma(q^{k-1})}. \]

We already know that

\[ \sigma(n^2) = q^k \cdot \left(\frac{D(n^2)}{\sigma(q^{k-1})}\right). \]

Since \( \sigma(q^k)\sigma(n^2) = 2^k n^2 \), we also obtain

\[ \frac{2n^2}{\sigma(q^k)} = \frac{\sigma(n^2)}{q^k} = \frac{D(n^2)}{\sigma(q^{k-1})}. \]

This implies that

\[ n^2 = \frac{\sigma(q^k)}{2} \cdot \left(\frac{D(n^2)}{\sigma(q^{k-1})}\right). \]

It follows that

\[ \gcd\left(n^2, \sigma(n^2)\right) = \frac{D(n^2)}{\sigma(q^{k-1})} \]

since

\[ \gcd\left(q^k, \frac{\sigma(q^k)}{2}\right) = \gcd(q^k, \sigma(q^k)) = 1. \]

This concludes the proof. \(\square\)

**Remark 2.1.** Dris obtained the lower bound 3 for \( \sigma(N/q^k)/q^k \) in [6] and [7]. The following papers obtain (ever-increasing) lower bounds for this quantity: [8, 4, 2, 5].

**Remark 2.2.** Notice that

\[ \frac{\sigma(n^2)}{q^k} = \frac{2n^2}{\sigma(q^k)} > \frac{8}{3} \cdot \left(\frac{n^2}{q^k}\right) \]

since \( I(q^k) < 5/4 \) holds unconditionally (i.e., for \( k \geq 1 \)). Additionally, note that

\[ \frac{8}{3} \cdot \left(\frac{n^2}{q^k}\right) > \frac{8n}{5} \]

is true if \( q^k < n \).

Dris conjectured in [6] that \( q^k < n \). Recently, Brown has announced a proof for \( q < n \), and that \( q^k < n \) holds “in many cases” [3].
**Remark 2.3.** It is an easy exercise to prove that \( q^k < n \) implies the biconditional

\[
q^k < n \iff \sigma(q^k) < \sigma(n) \iff \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}.
\]

We refer the interested reader to MSE (http://math.stackexchange.com/q/713035) for an expository proof.

Next, we prove the following lemmas.

**Lemma 2.2.** If \( N = q^k n^2 \) is an odd perfect number with Euler prime \( q \), then

\[
I(n^2) \geq 2 - \frac{5}{3q} \Rightarrow (k = 1 \land q = 5).
\]

**Proof.** Note that

\[
I(n^2) = \frac{2}{I(q^k)} = \frac{2q^k(q - 1)}{q^{k+1} - 1} = 2 - 2 \cdot \left( \frac{q^k - 1}{q^{k+1} - 1} \right).
\]

If

\[
I(n^2) \geq 2 - \frac{5}{3q},
\]

then we obtain

\[
2 - 2 \cdot \left( \frac{q^k - 1}{q^{k+1} - 1} \right) \geq 2 - \frac{5}{3q},
\]

\[
\frac{5}{3q} \geq 2 \cdot \left( \frac{q^k - 1}{q^{k+1} - 1} \right),
\]

\[
5q^{k+1} - 5 \geq 6q^{k+1} - 6q,
\]

\[
0 \geq q^{k+1} - 6q + 5,
\]

which then implies that \( k = 1 \). (Otherwise, if \( k > 1 \) we have

\[
0 \geq q^{k+1} - 6q + 5 \geq q^6 - 6q + 5,
\]

since \( k \equiv 1 \pmod{4} \), contradicting \( q \geq 5 \).) Now, since \( k = 1 \), we get

\[
0 \geq q^2 - 6q + 5 = (q - 5)(q - 1),
\]

which implies that \( 1 \leq q \leq 5 \). Together with \( q \geq 5 \), this means that \( q = 5 \). This concludes the proof.

**Lemma 2.3.** If \( N = q^k n^2 \) is an odd perfect number with Euler prime \( q \), then \( k = 1 \) implies

\[
I(n^2) \leq 2 - \frac{5}{3q}.
\]
Proof. Suppose that $k = 1$. By Lemma 1.1, we have $\sigma(n^2)/q \mid n^2$. This implies that there exists an (odd) integer $d$ such that
\[ n^2 = d \cdot \left( \frac{\sigma(n^2)}{q} \right). \]

Note that, from the equation $\sigma(N) = 2N$, we obtain (upon setting $k = 1$)
\[ (q + 1)\sigma(n^2) = \sigma(q)\sigma(n^2) = 2qn^2, \]
from which we get
\[ d = \frac{n^2}{\sigma(n^2)/q} = \frac{q + 1}{2}. \]

Notice that, when $k = 1$, we can derive
\[ \frac{5}{3} \leq I(n^2) = \frac{2}{I(q)} = \frac{2q}{q + 1} < 2, \]
so that we have
\[ \frac{q}{2} < d = \frac{q}{I(n^2)} \leq \frac{3q}{5}. \]

Additionally, note that, when $k = 1$, we have
\[ I(n^2) = \frac{2}{I(q)} = \frac{2q}{q + 1} = \frac{2q + 2}{q + 1} - \frac{2}{q + 1} = 2 - \frac{1}{q + 1} = 2 - \frac{1}{d}. \]

Consequently, we obtain
\[ \frac{q}{2} < d \leq \frac{3q}{5} \]
\[ \frac{5}{3q} \leq \frac{1}{d} < \frac{2}{q} \]
\[ 2 - \frac{2}{q} < 2 - \frac{1}{d} = I(n^2) \leq 2 - \frac{5}{3q}, \]
and we are done. \[ \square \]

3 The proof of Lemma 1.1

Let $N = q^k n^2$ be an odd perfect number with Euler prime $q$.

By Lemma 2.1, we have
\[ \frac{D(n^2)}{\sigma(q^{k-1})} = \frac{\sigma(N/q^k)}{q^k}. \]

This equation can be rewritten as
\[ D(n^2) = \frac{\sigma(n^2)}{q} \cdot I(q^{k-1}). \]

Suppose that $\sigma(n^2)/q \mid n^2$. Trivially, we know that $\sigma(n^2)/q \mid \sigma(n^2)$. Thus, we have
\[ \frac{\sigma(n^2)}{q} \mid (2n^2 - \sigma(n^2)) = D(n^2), \]

giving
\[
\frac{\sigma(n^2)}{q} \mid \frac{\sigma(n^2)}{q} \cdot I(q^{k-1}).
\]
This implies that \( I(q^{k-1}) \) is an integer. Since \( 1 \leq I(q^{k-1}) < 5/4 \), we obtain \( k = 1 \).

Now assume that \( k = 1 \). We obtain
\[
2n^2 - \sigma(n^2) = D(n^2) = \frac{\sigma(n^2)}{q}.
\]
Again, since \( \sigma(n^2)/q \mid \sigma(n^2) \), this implies
\[
\frac{\sigma(n^2)}{q} \mid n^2
\]
since \( \sigma(n^2)/q \) is odd.

This concludes the proof of Lemma 1.1.

4 The proof of Lemma 1.2

Let \( N = q^n n^2 \) be an odd perfect number with Euler prime \( q \).

Assume to the contrary that
\[
I(n^2) > 2 - \frac{5}{3q}.
\]
Following the proof of Lemma 2.2, we get
\[
0 > q^{k+1} - 6q + 5.
\]
Since \( k \equiv 1 \pmod{4} \), then \( k \geq 1 \), which implies that
\[
0 > q^{k+1} - 6q + 5 \geq q^2 - 6q + 5 = (q-5)(q-1).
\]
This then finally gives \( 1 < q < 5 \), contradicting \( q \geq 5 \).

We therefore conclude that
\[
I(n^2) \leq 2 - \frac{5}{3q},
\]
and this finishes the proof of Lemma 1.2.

5 The proof of Lemma 1.3

Let \( N = q^n n^2 \) be an odd perfect number with Euler prime \( q \).

By Lemma 2.1, we have
\[
\frac{D(n^2)}{\sigma(q^{k-1})} = \frac{2n^2}{\sigma(q^{k})}.
\]
Multiplying throughout the last equation by \( \sigma(q^{k-1})\sigma(q^{k}) \), we get
\[
D(n^2)\sigma(q^k) = 2n^2\sigma(q^{k-1}).
\]
If \( k = 1 \), then it is evident that \( D(n^2) \mid 2n^2 \), from which it follows that \( D(n^2) \mid n^2 \), since \( D(n^2) \) is odd.

Now, assume that \( D(n^2) \mid n^2 \). Then we have

\[
\frac{\sigma(q^k)}{2\sigma(q^{k-1})} = \frac{n^2}{D(n^2)}
\]

is an integer. Since \( \gcd(\sigma(q^{k-1}), \sigma(q^k)) = 1 \), the previous equation then implies that \( k = 1 \).

This concludes the proof of Lemma 1.3. In particular, we have shown that the Descartes–Frenicle–Sorli conjecture for odd perfect numbers \( q^k n^2 \) is true if and only if the non-Euler part \( n^2 \) is deficient-perfect [12].

6 The proof of Theorem 1.1

Let \( N = q^k n^2 \) be an odd perfect number with Euler prime \( q \).

We want to prove that the equation

\[
I(n^2) = 2 - \frac{5}{3q}
\]

holds if and only if \( k = 1 \) and \( q = 5 \).

Suppose that

\[
I(n^2) = 2 - \frac{5}{3q}.
\]

Following the proof of Lemma 2.1, we get

\[
0 = q^{k+1} - 6q + 5.
\]

Assume to the contrary that \( k > 1 \). Since \( k \equiv 1 \pmod{4} \), we obtain

\[
0 = q^{k+1} - 6q + 5 \geq q^6 - 6q + 5.
\]

This contradicts \( q \geq 5 \). Thus, we have established that \( k = 1 \).

Substituting \( k = 1 \) into \( 0 = q^{k+1} - 6q + 5 \), we have

\[
0 = q^2 - 6q + 5 = (q - 5)(q - 1)
\]

which implies that \( q = 5 \) since \( q \geq 5 \). This takes care of one direction of Theorem 1.1.

For the other direction, assume that \( k = 1 \) and \( q = 5 \). We want to show that

\[
I(n^2) = 2 - \frac{5}{3q}.
\]

Note that, when \( k = 1 \) and \( q = 5 \), we obtain

\[
I(n^2) = \frac{2}{I(q)} = \frac{2q}{q+1} = \frac{5}{3}.
\]

We also get

\[
2 - \frac{5}{3q} = 2 - \frac{1}{3} = \frac{5}{3},
\]

so that we have

\[
I(n^2) = 2 - \frac{5}{3q},
\]

as desired.
7 Concluding remarks

We end with some remarks related to the biconditional

\[ k = 1 \iff \left( D(n^2) \mid n^2 \right). \]

Suppose that \( k = 1 \). By Lemma 1.3 and Lemma 2.1, we obtain

\[ D(n^2) = \gcd(n^2, \sigma(n^2)) = \frac{\sigma(n^2)}{q} = \frac{n^2}{(q + 1)/2}. \]

Multiplying throughout the equations by \( q(q + 1)/2 \), we have

\[ D(n^2) \cdot \left( \frac{q(q + 1)}{2} \right) = \left( \frac{q + 1}{2} \right) \cdot \sigma(n^2) = qn^2 = N. \]

In fact, as shown by Slowak [13], every odd perfect number \( N \) has the form

\[ N = q^k \cdot \frac{\sigma(q^k)}{2} \cdot d \]

for some \( d > 1 \). We give a quick proof of this fact here.

By Lemma 2.1, we obtain

\[ \frac{D(n^2)}{\sigma(q^k)} = \gcd(n^2, \sigma(n^2)) = \frac{\sigma(n^2)}{q^k} = \frac{n^2}{\sigma(q^k)/2}. \]

Multiplying throughout the equations by \( q^k \sigma(q^k)/2 \), we get

\[ \frac{q^k \sigma(q^k)}{2} \cdot D(n^2) = \frac{q^k \sigma(q^k)}{2} \cdot \gcd(n^2, \sigma(n^2)) = q^k n^2 = N, \]

where

\[ d = \frac{D(n^2)}{\sigma(q^k)} = \gcd(n^2, \sigma(n^2)) > 1 \]

by Remark 2.1.

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