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Some enumerations of non-trivial composition of the differential operations and the directional derivative

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Abstract: This paper deals with some enumerations of the higher order non-trivial compositions of the differential operations and the directional derivative in the space $\mathbb{R}^n$ $(n\geq 3)$. One new enumeration of the higher order non-trivial compositions is obtained.
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1 The enumeration of the higher order non-trivial compositions of the differential operations and the directional derivative in the space $\mathbb{R}^3$

Consider the sets of the smooth functions

$$\mathrm{A}_0=\big\{f:\mathbb{R}^3\to\mathbb{R}\,|\,f\in C^\infty(\mathbb{R}^3)\big\} \text{ and } \mathrm{A}_1=\big\{\vec{f}:\mathbb{R}^3\to\mathbb{R}^3\,|\,f_1,f_2,f_3\in C^\infty(\mathbb{R}^3)\big\}$$

in the three-dimensional Euclidean space $\mathbb{R}^3$. Let $\vec{e}=(e_1,e_2,e_3)\in\mathbb{R}^3$ be a unit vector. The gradient, curl, divergence, and the Gateaux directional derivative in a direction $\vec{e}$ are defined in the terms of the partial derivative operators as follows:

$$
\begin{aligned}
\operatorname{grad} f &= \nabla_1 f = \tfrac{\partial f}{\partial x_1}\,\vec{i}+\tfrac{\partial f}{\partial x_2}\,\vec{j}+\tfrac{\partial f}{\partial x_3}\,\vec{k},\ \nabla_1 : \mathrm{A}_0 \longrightarrow \mathrm{A}_1\,,\\
\operatorname{curl} \vec{f} &= \nabla_2 \vec{f} = \left(\tfrac{\partial f_3}{\partial x_2}-\tfrac{\partial f_2}{\partial x_3}\right)\vec{i}+\left(\tfrac{\partial f_1}{\partial x_3}-\tfrac{\partial f_3}{\partial x_1}\right)\vec{j}+\left(\tfrac{\partial f_2}{\partial x_1}-\tfrac{\partial f_1}{\partial x_2}\right)\vec{k},\ \nabla_2 : \mathrm{A}_1 \longrightarrow \mathrm{A}_1\,,\\
\operatorname{div} \vec{f} &= \nabla_3 \vec{f} = \tfrac{\partial f_1}{\partial x_1}+\tfrac{\partial f_2}{\partial x_2}+\tfrac{\partial f_3}{\partial x_3},\ \nabla_3 : \mathrm{A}_1 \longrightarrow \mathrm{A}_0\,,\\
\operatorname{dir}_{\vec{e}} f &= \nabla_0 f = \nabla_1 f \cdot \vec{e} = \tfrac{\partial f}{\partial x_1}\,e_1+\tfrac{\partial f}{\partial x_2}\,e_2+\tfrac{\partial f}{\partial x_3}\,e_3,\ \nabla_0 : \mathrm{A}_0 \longrightarrow \mathrm{A}_0\,.
\end{aligned}
$$

Let $\mathcal{A}_3 = \{\nabla_1, \nabla_2, \nabla_3\}$ and $\mathcal{B}_3 = \{\nabla_0, \nabla_1, \nabla_2, \nabla_3\}$. Malešević [12] has proved that the number of the k^{th} order compositions over the set $\mathcal{A}_3$ is $\mathbf{f}(k) = F_{k+3}$, F_k is the k^{th} Fibonacci number. A composition of differential operations that is not equal to 0 or $\vec{0}$ is called non-trivial. Malešević [11] has showed that the number of the k^{th} order non-trivial compositions over the set $\mathcal{A}_3$ is $\mathbf{g}(k) = 3$. Schreiber [17, Section 5.2] has listed the higher order non-trivial compositions over the set $\mathcal{A}_3$:

$$
\begin{aligned}
\text{(grad) div} \ldots \text{ grad div grad } f &= (\nabla_1 \circ)\nabla_3 \circ \cdots \circ \nabla_1 \circ \nabla_3 \circ \nabla_1 f,\\
\text{curl curl} \ldots \text{ curl curl curl } \vec{f} &= \nabla_2 \circ \nabla_2 \circ \cdots \circ \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f},\\
\text{(div) grad} \ldots \text{ div grad div } \vec{f} &= (\nabla_3 \circ)\nabla_1 \circ \cdots \circ \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f}.
\end{aligned}
$$

The terms in brackets are included if the number of the differential operations is odd and are left out otherwise. Malešević, and Jovović [14] have proved that the number of the k^{th} order compositions over the set $\mathcal{B}_3$ is $\mathbf{f}^{\mathrm{G}}(k) = 2^{k+1}$.

According to the above results, it is natural to try to calculate the number of the non-trivial compositions over the set $\mathcal{B}_3$. A straightforward verification shows that all compositions of the second order over $\mathcal{B}_3$ are

$$
\begin{aligned}
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_0 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_1 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big),\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_0 \circ \nabla_3 \vec{f} = \big(\nabla_1 \circ \nabla_3 \vec{f}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{div} \vec{f} = \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{div} \operatorname{grad} f = \nabla_3 \circ \nabla_1 f = \Delta f,\\
&\operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \vec{f} = 0,
\end{aligned}
$$

and that only the last two are trivial. This fact leads us to use the following method for determining the number of the non-trivial compositions over the set $\mathcal{B}_3$. We define a binary relation σ on the set $\mathcal{B}_3$ as follows:

$$\nabla_i \,\sigma\, \nabla_j \text{ iff the composition } \nabla_j \circ \nabla_i \text{ is non-trivial.}$$

The relation σ induces the Cayley table

σ	∇_0	∇_1	∇_2	∇_3
∇_0	1	1	0	0
∇_1	0	0	0	1
∇_2	0	0	1	0
∇_3	1	1	0	0

For a convenience, we extend the set $\mathcal{B}_3$ with the nowhere-defined function ∇_{-1}, whose domain and range are empty sets, and establish $\nabla_{-1}\,\sigma\,\nabla_i$ $(0 \leq i \leq 3)$. Thus, the graph Γ of the relation σ is rooted tree with a root ∇_{-1}.

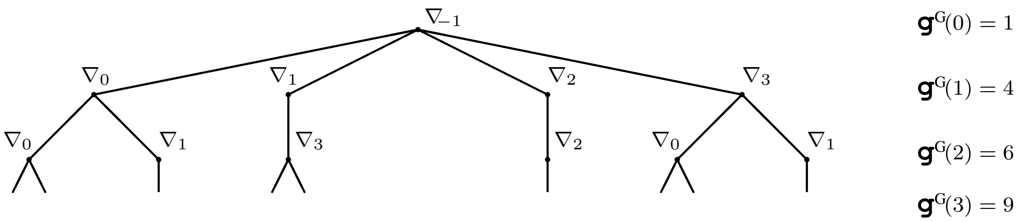

Figure 1. Tree Γ

Here we would like to point out, that the child of ∇_i is ∇_j if composition $\nabla_j \circ \nabla_i$ is non-trivial. For any non-trivial composition $\nabla_{i_k} \circ \cdots \circ \nabla_{i_1}$ there is a unique path in the tree Γ, such that the level of vertex ∇_{i_j} is j $(1 \leq j \leq k)$.

Let $\mathsf{g}^{\mathrm{G}}(k)$ be the number of the k^{th} order non-trivial compositions over the set $\mathcal{B}_3$. Let $\mathsf{g}_i^{\mathrm{G}}(k)$ be the number of the k^{th} order non-trivial compositions starting with ∇_i. Then we have $\mathsf{g}^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k) + \mathsf{g}_1^{\mathrm{G}}(k) + \mathsf{g}_2^{\mathrm{G}}(k) + \mathsf{g}_3^{\mathrm{G}}(k)$. We can also obtain the equalities $\mathsf{g}_0^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1)$, $\mathsf{g}_1^{\mathrm{G}}(k) = \mathsf{g}_3^{\mathrm{G}}(k-1)$, $\mathsf{g}_2^{\mathrm{G}}(k) = \mathsf{g}_2^{\mathrm{G}}(k-1)$, $\mathsf{g}_3^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1)$. Since the only child of ∇_2 is ∇_2, we can deduce $\mathsf{g}_2^{\mathrm{G}}(k) = \mathsf{g}_2^{\mathrm{G}}(k-1) = \cdots = \mathsf{g}_2^{\mathrm{G}}(1) = 1$. Putting things together we obtain the recurrence for $\mathsf{g}^{\mathrm{G}}(k)$:

$$\begin{aligned}
\mathsf{g}^{\mathrm{G}}(k) &= \mathsf{g}_0^{\mathrm{G}}(k) + \mathsf{g}_1^{\mathrm{G}}(k) + \mathsf{g}_2^{\mathrm{G}}(k) + \mathsf{g}_3^{\mathrm{G}}(k) \\
&= \big(\mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1)\big) + \mathsf{g}_3^{\mathrm{G}}(k-1) + \mathsf{g}_2^{\mathrm{G}}(k-1) + \big(\mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1)\big) \\
&= \mathsf{g}^{\mathrm{G}}(k-1) + \mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1) \\
&= \mathsf{g}^{\mathrm{G}}(k-1) + \big(\mathsf{g}_0^{\mathrm{G}}(k-2) + \mathsf{g}_1^{\mathrm{G}}(k-2)\big) + \mathsf{g}_3^{\mathrm{G}}(k-2) + \mathsf{g}_2^{\mathrm{G}}(k-2) - \mathsf{g}_2^{\mathrm{G}}(k-2) \\
&= \mathsf{g}^{\mathrm{G}}(k-1) + \mathsf{g}^{\mathrm{G}}(k-2) - 1.
\end{aligned}$$

Substituting $\mathtt{t}(k) = \mathsf{g}^{\mathrm{G}}(k) - 1$ into the previous formula we obtain the recurrence $\mathtt{t}(k) = \mathtt{t}(k-1) + \mathtt{t}(k-2)$. Based on the initial conditions $\mathsf{g}^{\mathrm{G}}(1) = 4$ and $\mathsf{g}^{\mathrm{G}}(2) = 6$, i.e., $\mathtt{t}(1) = 3$ and $\mathtt{t}(2) = 5$, we conclude that $\mathsf{g}^{\mathrm{G}}(k) = F_{k+3} + 1$.

2 The enumerations of the higher order non-trivial compositions of the differential operations and the directional derivative in the space $\mathbb{R}^n$

We start this section by recalling some definitions of the theory of differential forms. Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space ($n\geq 3$) and consider the set of smooth functions

$$\mathrm{A}_0 = \{f:\mathbb{R}^n\to\mathbb{R} \mid f\in C^\infty(\mathbb{R}^n)\}.$$

The set of all differential k-forms on $\mathbb{R}^n$, denoted by $\Omega^k(\mathbb{R}^n)$, is a free A_0-module of the rank $\binom{n}{k}$ with the standard basis $\{dx_I = dx_{i_1}\cdots dx_{i_k} \mid 1\leq i_1<\cdots<i_k\leq n\}$. A differential k-form ω can be written uniquely as $\omega=\sum_{I\in\mathcal{I}}\omega_I dx_I$, where $\omega_I\in \mathrm{A}_0$, and $\mathcal{I}=\mathcal{I}(k,n)$ is the set of multi-indices $I=(i_1,\ldots,i_k)$, $(1\leq i_1<\cdots<i_k\leq n)$. The complement of I is $J=(j_1,\ldots,j_{n-k})\in\mathcal{I}(n-k,n)$, $(1\leq j_1<\cdots<j_{n-k}\leq n)$, where components j_p are the elements of the set $\{1,\ldots,n\}\backslash\{i_1,\ldots,i_k\}$. We have $dx_I dx_J = \sigma(I)dx_1\ldots dx_n$, where $\sigma(I)$ is a signature of the permutation $(i_1,\ldots,i_k, j_1,\ldots,j_{n-k})$. Note that $\sigma(J)=(-1)^{k(n-k)}\sigma(I)$. With the notion mentioned above we define $\star_k(dx_I)=\sigma(I)dx_J$. A map $\star_k:\Omega^k(\mathbb{R}^n)\longrightarrow\Omega^{n-k}(\mathbb{R}^n)$ defined by $\star_k(\omega)=\sum_{I\in\mathcal{I}(k,n)}\omega_I \star_k(dx_I)$ is the Hodge star operator and it provides a natural isomorphism between $\Omega^k(\mathbb{R}^n)$ and $\Omega^{n-k}(\mathbb{R}^n)$. The Hodge star operator applied twice to a differential k-form yields $\star_{n-k}(\star_k\omega)=(-1)^{nk+k+s}(\omega)$, where s is the number of negative signs in the inner product of the base vectors of the space $\mathbb{R}^n$ (see [2], p. 29). For the inverse of $\star_k$ the equality ${\star_k}^{-1}(\psi)=(-1)^{nk+k+s}\star_{n-k}(\psi)$ holds, where $\psi\in\Omega^{n-k}(\mathbb{R}^n)$.

A differential 0-form is a function $f(x_1,\ldots,x_n)\in \mathrm{A}_0$. We define df to be the differential 1-form $df=\sum_{i=1}^n\frac{\partial f}{\partial x_i}dx_i$. Given a differential k-form $\sum_{I\in\mathcal{I}}\omega_I dx_I$, the exterior derivative $d_k\omega$ is the differential $(k+1)$-form $d_k\omega=\sum_{I\in\mathcal{I}}d\omega_I dx_I$. The exterior derivative d_k is a linear map $d_k:\Omega^k(\mathbb{R}^n)\longrightarrow\Omega^{k+1}(\mathbb{R}^n)$ which obeys Leibnitz rule

$$d_{p+q}(\omega\,\psi)=(d_p\omega)\,\psi+(-1)^p\omega\,(d_q\psi),$$

ω and ψ are differential p-form and q-form. The exterior derivative has a property

$$d_{k+1}(d_k\omega)=0,$$

for any differential k-form ω. For more details on the topic please refer to [2, 16, 22]. Some historical notes on the development of the theory of differential forms are given in [7, 21].

Consider the sets of the smooth functions

$$\mathrm{A}_k=\big\{\vec{f}:\mathbb{R}^n\to\mathbb{R}^{\binom{n}{k}} \mid f_1,\ldots,f_{\binom{n}{k}}\in C^\infty(\mathbb{R}^n)\big\}\quad\big(m=\Big\lfloor\frac{n}{2}\Big\rfloor,\,0\leq k\leq m\big).$$

Let $p_k : \Omega^k(\mathbb{R}^n) \to \mathrm{A}_k$ be a presentation of the differential form in coordinate notation. Define functions φ_i $(0 \le i \le m)$ and φ_{n-j} $(0 \le j < n-m)$ as follows:

$$\varphi_i = p_i : \Omega^i(\mathbb{R}^n) \longrightarrow \mathrm{A}_i \quad \text{and} \quad \varphi_{n-j} = p_j \star_j^{-1} : \Omega^{n-j}(\mathbb{R}^n) \longrightarrow \mathrm{A}_j.$$

Analogously to Malešević [13], the combination of the Hodge star operator and the exterior derivative generates differential operations:

$$\nabla_k = \varphi_k \, d_{k-1} \, \varphi_{k-1}^{-1} \quad (1 \le k \le n).$$

See the list below.

$\mathcal{A}_n$ $(n=2m)$:

$\nabla_1 = p_1 \, d_0 \, p_0^{-1} : A_0 \to A_1$
$\nabla_2 = p_2 \, d_1 \, p_1^{-1} : A_1 \to A_2$
$\vdots$
$\nabla_i = p_i \, d_{i-1} \, p_{i-1}^{-1} : A_{i-1} \to A_i$
$\vdots$
$\nabla_m = p_m \, d_{m-1} \, p_{m-1}^{-1} : A_{m-1} \to A_m$
$\nabla_{m+1} = p_{m-1} \star_{m-1}^{-1} d_m \, p_m^{-1} : A_m \to A_{m-1}$
$\nabla_{m+2} = p_{m-2} \star_{m-2}^{-1} d_{m+1} \star_{m-1} p_{m-1}^{-1} : A_{m-1} \to A_{m-2}$
$\vdots$
$\nabla_{n-j} = p_j \star_j^{-1} d_{n-(j+1)} \star_{j+1} p_{j+1}^{-1} : A_{j+1} \to A_j$
$\vdots$
$\nabla_{n-1} = p_1 \star_1^{-1} d_{n-2} \star_2 p_2^{-1} : A_2 \to A_1$
$\nabla_n = p_0 \star_0^{-1} d_{n-1} \star_1 p_1^{-1} : A_1 \to A_0$,

$\mathcal{A}_n$ $(n=2m+1)$:

$\nabla_1 = p_1 \, d_0 \, p_0^{-1} : A_0 \to A_1$
$\nabla_2 = p_2 \, d_1 \, p_1^{-1} : A_1 \to A_2$
$\vdots$
$\nabla_i = p_i \, d_{i-1} \, p_{i-1}^{-1} : A_{i-1} \to A_i$
$\vdots$
$\nabla_m = p_m \, d_{m-1} \, p_{m-1}^{-1} : A_{m-1} \to A_m$
$\nabla_{m+1} = p_m \star_m^{-1} d_m \, p_m^{-1} : A_m \to A_m$
$\nabla_{m+2} = p_{m-1} \star_{m-1}^{-1} d_{m+1} \star_m p_m^{-1} : A_m \to A_{m-1}$
$\nabla_{m+3} = p_{m-2} \star_{m-2}^{-1} d_{m+2} \star_{m-1} p_{m-1}^{-1} : A_{m-1} \to A_{m-2}$
$\vdots$
$\nabla_{n-j} = p_j \star_j^{-1} d_{n-(j+1)} \star_{j+1} p_{j+1}^{-1} : A_{j+1} \to A_j$
$\vdots$
$\nabla_{n-1} = p_1 \star_1^{-1} d_{n-2} \star_2 p_2^{-1} : A_2 \to A_1$
$\nabla_n = p_0 \star_0^{-1} d_{n-1} \star_1 p_1^{-1} : A_1 \to A_0$.

Table 1. List of differential operations in $\mathbb{R}^n$

For $n=3$, we obtain the standard definitions of the gradient, curl and divergence, see [2, 8]. For $n=4$ let us consider Lewis–Wilson four-dimensional differential operations [24]:

$$\begin{aligned}
\nabla_1(f) &= \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, -\frac{\partial f}{\partial x_4}\Big) : A_0 \longrightarrow A_1, \\
\nabla_2(\mathbf{f}) &= \nabla_2\big((f_1, f_2, f_3, f_4)\big) \\
&= \Big(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}, \frac{\partial f_4}{\partial x_1} + \frac{\partial f_1}{\partial x_4}, \frac{\partial f_4}{\partial x_2} + \frac{\partial f_2}{\partial x_4}, \frac{\partial f_4}{\partial x_3} + \frac{\partial f_3}{\partial x_4}\Big) : A_1 \longrightarrow A_2, \\
\nabla_3(\mathbf{F}) &= \nabla_3\big((F_1, F_2, F_3, F_4, F_5, F_6)\big) \\
&= \Big(\frac{\partial F_6}{\partial x_2} - \frac{\partial F_5}{\partial x_3} - \frac{\partial F_1}{\partial x_4}, \frac{\partial F_4}{\partial x_3} - \frac{\partial F_6}{\partial x_1} - \frac{\partial F_2}{\partial x_4}, \frac{\partial F_5}{\partial x_1} - \frac{\partial F_4}{\partial x_2} - \frac{\partial F_3}{\partial x_4}, \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}\Big) : A_2 \longrightarrow A_1, \\
\nabla_4(\mathbf{f}) &= \nabla_4\big((f_1, f_2, f_3, f_4)\big) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4} : A_1 \longrightarrow A_0,
\end{aligned}$$

over the set $\mathcal{A}_4 = \{\nabla_1, \nabla_2, \nabla_3, \nabla_4\}$. Using Lewis–Wilson determination of the Hodge operators on the base vectors (see [24], p. 450), we may conclude that Lewis–Wilson determination of four-dimensional differential operations is consentient with previous definition of the differential operations in the space $\mathbb{R}^4$. Therefore, $\nabla_2 \circ \nabla_1 = 0$ and $\nabla_3 \circ \nabla_2 = 0$ and $\nabla_4 \circ \nabla_3 = 0$ are true. Let us emphasize that, analogously to results of F.C. Chang [5, 6], previous four-dimensional differential operations could be determined by the appropriate matrix formulation.

Let us note that the formulas for determination a number of the compositions over the set $\mathcal{A}_n$ and the corresponding recurrences are obtained by Malešević [12, 13]. An application of the formulas is given by Myers [15]. The corresponding integer sequences can be found in [18]. For the proofs of the following two theorems we refer the reader to [12].

Theorem 2.1. *A non-trivial composition over the set $\mathcal{A}_n$ is of the form:*

$$(\nabla_i \circ)\, \nabla_{n+1-i} \circ \nabla_i \circ \cdots \circ \nabla_{n+1-i} \circ \nabla_i,$$

for some i $(2i,\ 2i-2 \neq n,\ 1 \leq i \leq n)$. The term in brackets is included if the number of the differential operations is odd and is left out otherwise.

In the terms of the Hodge star operator and the exterior derivative, we have the following representation of the non-trivial composition

$$(\nabla_i \circ)\, \nabla_{n+1-i} \circ \nabla_i \circ \cdots \circ \nabla_{n+1-i} \circ \nabla_i =$$
$$\begin{cases} (\,p_i\, d_{i-1}\, p_{i-1}^{-1}\,)\, p_{i-1} \star_{i-1}^{-1} d_{n-i} \star_i d_i \cdots \star_{i-1}^{-1} d_{n-i} \star_i d_{i-1}\, p_{i-1}, & i \leq m; \\ (\,p_{n-i} \star_{n-i}^{-1} d_{i-1} \star_{n+1-i} p_{n+1-i}^{-1}\,)\, p_{n+1-i}\, d_{n-i} \star_{n-i}^{-1} d_{i-1} \star_{n+1-i} & \\ \cdots d_{n-i} \star_{n-i}^{-1} d_{i-1} \star_{n+1-i} p_{n+1-i}^{-1}, & i > m; \end{cases}$$

where $i \in \{1, \ldots, n\} \backslash \{m, m+1\}$ if $n = 2m$ or $i \in \{1, \ldots, n\}$ if $n = 2m+1$.

Theorem 2.2. *Let $\mathrm{g}(k)$ be the number of the k^{th} order non-trivial compositions over the set $\mathcal{A}_n$. Then we have*

$$\mathrm{g}(k) = \begin{cases} n, & 2 \nmid n\,; \\ n, & 2 \,|\, n\ ,\ k = 1\,; \\ n-1, & 2 \,|\, n\ ,\ k = 2\,; \\ n-2, & 2 \,|\, n\ ,\ k > 2\,. \end{cases}$$

The Hodge dual to the exterior derivative $d_k : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ is codifferential δ_{k-1}. It is a generalization of the divergence. The codifferential is a linear map $\delta_{k-1} : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$, determined by

$$\delta_{k-1} = (-1)^{nk+k+s+1} \star_{n-(k-1)} d_{n-k} \star_k = (-1)^k \star_{k-1}^{-1} d_{n-k} \star_k$$

(see [2], p. 33). Note that $\nabla_{n-j} = (-1)^{j+1}\, p_j\ \delta_j\ p_{j+1}^{-1}$, $(0 \leq j < n-m-1)$. The codifferential can be coupled with the exterior derivative to construct the Hodge Laplacian, also known as the Laplace-de Rham operator, $\Delta_k : \Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$. The Hodge Laplacian is a harmonic generalization of the Laplace differential operator, given by $\Delta_0 = \delta_0 d_0$ and $\Delta_k = \delta_k d_k + d_{k-1}\delta_{k-1}$, for $1 \leq k \leq m$, see [23]. The operator Δ_0 is actually the negative of the Laplace-Beltrami (scalar) operator.

A k-form ω is called harmonic if $\Delta_k(\omega) = 0$. We say that $\vec{f} \in A_k$ is a harmonic function if $\omega = {p_k}^{-1}(\vec{f})$ is a harmonic k-form. If $k \geq 1$ harmonic function $\vec{f}$ is also called a harmonic field. For the function $\vec{f} \in A_k$ $(1 \leq k \leq m)$ we have that $\Delta_k(p_k^{-1}\vec{f}) = 0$ iff $\delta_{k-1}(p_k^{-1}\vec{f}) = 0$ and $d_k(p_k^{-1}\vec{f}) = 0$, [23, Proposition 4.15]. In fact, we obtain the following lemma.

Lemma 2.3. *Let* $\vec{f} \in A_k$ $(1 \leq k \leq m)$*. Then*

$$\Delta_k(p_k^{-1}\vec{f}) = 0 \iff \nabla_{n-(k-1)}(\vec{f}) = 0 \wedge \nabla_{k+1}(\vec{f}) = 0.$$

For harmonic function $f \in A_0$ we have $\Delta_0 f = \delta_0 d_0 f = 0$. Hence we get $\nabla_n \circ \nabla_1 f = 0$ and, consequently, we obtain $(\nabla_1 \circ)\nabla_n \circ \nabla_1 \circ \cdots \circ \nabla_n \circ \nabla_1 f = 0$. We can now rephrase Theorem 2.1 for harmonic functions.

Theorem 2.4. *All the second and higher order non-trivial compositions over the set $\mathcal{A}_n$ acting on harmonic function $f \in A_0$ are trivial. Furthermore, all the first and higher order non-trivial compositions over the set $\mathcal{A}_n$ acting on harmonic field $\vec{f} \in A_k$ $(1 \leq k \leq m)$ are trivial.*

We say that $\vec{f} \in A_k$ $(1 \leq k \leq m)$ is a coordinate-harmonic function or that $\vec{f}$ satisfies harmonic coordinate condition, if all its coordinates are harmonic functions. Malešević [11] showed that all the third and higher order non-trivial compositions of the differential operations acting on coordinate-harmonic functions are trivial in $\mathbb{R}^3$.

Conjecture 2.5. *All the third and higher order non-trivial compositions over the set $\mathcal{A}_n$ acting on coordinate-harmonic functions are trivial in $\mathbb{R}^n$.*

Lewis and Wilson [10, 24] gave an approach to a coordinate investigation of Conjecture 2.5 for $n = 4$, see also [19, 20, 9]. A similar problem for coordinate-harmonic functions can be found in discrete exterior calculus [4] and combinatorial Hodge theory [1]. Remark that some applications of directional derivatives in discrete approximations of higher order differential operations are considered in [3].

Let $f \in \mathrm{A}_0$ be a scalar function, and $\vec{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$ be a unit vector. The Gateaux directional derivative in a direction $\vec{e}$ is defined by

$$\mathrm{dir}_{\vec{e}}\, f = \nabla_0 f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} e_k : A_0 \longrightarrow A_0\,.$$

Extend a set of differential operations $\mathcal{A}_n = \{\nabla_1, \ldots, \nabla_n\}$ with the directional derivative ∇_0 to the set $\mathcal{B}_n = \mathcal{A}_n \cup \{\nabla_0\} = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$. Malešević, and Jovović [14] have given the recurrences for counting the number of compositions over the set $\mathcal{B}_n$. The corresponding integer sequences can be find in [18].

Consider the non-trivial compositions over the set $\mathcal{B}_n$ containing ∇_0.

First of all, we can compose the directional derivative ∇_0 by itself to obtain the non-trivial compositions. In other words, the non-trivial compositions containing just ∇_0 are obtained from

∇_0 by substituting $\nabla_0 \mapsto \nabla_0 \circ \nabla_0$. We define the coordinate Gateaux directional derivative by $\boldsymbol{\nabla}_0(\vec{f}) = (\nabla_0(f_1), \ldots, \nabla_0(f_n)) : A_1 \longrightarrow A_1$, for the unit vector $\vec{e}$. The following two equalities

$$\nabla_1 \circ \nabla_0 = \boldsymbol{\nabla}_0 \circ \nabla_1 \qquad \text{and} \qquad \nabla_0 \circ \nabla_n = \nabla_n \circ \boldsymbol{\nabla}_0$$

hold. Therefore, the non-trivial compositions over the set $\mathcal{B}_n$ containing ∇_0 can be obtained from $(\,\nabla_1\circ) \nabla_n \circ \nabla_1 \circ \cdots \circ \nabla_n \circ \nabla_1(\circ\nabla_n)$ by substituting $\nabla_1 \mapsto \nabla_1 \circ \nabla_0$ or $\nabla_n \mapsto \nabla_0 \circ \nabla_n$. Summarizing these facts, we have the following theorem.

Theorem 2.6. *A non-trivial composition over the set $\mathcal{B}_n$ has the one of the following forms:*

(i) $\nabla_0^k = \underbrace{\nabla_0 \circ \nabla_0 \circ \cdots \circ \nabla_0}_{k} \quad (k \in \mathbb{N})$;

(ii) $(\,\nabla_i \circ) \nabla_{n+1-i} \circ \nabla_i \circ \cdots \circ \nabla_{n+1-i} \circ \nabla_i \quad (2i,\, 2i-2 \neq n,\ 1 \leq i \leq n)$;

(iii) $(\,\nabla_1\circ) \nabla_0^{k_p} \circ \nabla_n \circ \nabla_1 \circ \cdots \circ \nabla_0^{k_2} \circ \nabla_n \circ \nabla_1 \circ \nabla_0^{k_1} \quad (k_1, \ldots, k_p \in \mathbb{N} \cup \{0\})$;

(iv) $(\,\nabla_0^{k_q} \circ \nabla_n\circ) \nabla_1 \circ \nabla_0^{k_{q-1}} \circ \nabla_n \circ \cdots \circ \nabla_1 \circ \nabla_0^{k_1} \circ \nabla_n \quad (k_1, \ldots, k_q \in \mathbb{N} \cup \{0\})$.

The terms in brackets are included if the number of the differential operations is odd and is left out otherwise.

The number of the higher order non-trivial compositions over the set $\mathcal{B}_n$ is determined by the binary relation ν, defined by:

$$\nabla_i \nu \nabla_j \text{ iff } (i=0 \wedge j=0) \vee (i=0 \wedge j=1) \vee (i=n \wedge j=0) \vee (i+j=n+1 \wedge 2i \neq n).$$

By applying Theorem 2.2 we conclude that the number of the k^{th} order non-trivial compositions starting with $\nabla_2, \ldots, \nabla_{n-1}$ can be expressed by formula

$$\mathsf{j}(k) = \mathsf{g}(k) - 2 = \begin{cases} n-2, & 2 \nmid n\,; \\ n-2, & 2 \mid n\ ,\ k=1\,; \\ n-3, & 2 \mid n\ ,\ k=2\,; \\ n-4, & 2 \mid n\ ,\ k>2\,. \end{cases}$$

Let $\mathsf{g}^{\mathrm{G}}(k)$ be the number of the k^{th} order non-trivial compositions over the set $\mathcal{B}_n$. Let $\mathsf{g}_0^{\mathrm{G}}(k)$, $\mathsf{g}_1^{\mathrm{G}}(k)$ and $\mathsf{g}_n^{\mathrm{G}}(k)$ be the numbers of the k^{th} order non-trivial compositions starting with ∇_0, ∇_1 and ∇_n, respectively. Then we have $\mathsf{g}^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k) + \mathsf{g}_1^{\mathrm{G}}(k) + \mathsf{j}(k) + \mathsf{g}_n^{\mathrm{G}}(k)$. Denote $\widetilde{\mathsf{g}}^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k) + \mathsf{g}_1^{\mathrm{G}}(k) + \mathsf{g}_n^{\mathrm{G}}(k)$. The following three recurrences are true $\mathsf{g}_0^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1)$, $\mathsf{g}_1^{\mathrm{G}}(k) = \mathsf{g}_n^{\mathrm{G}}(k-1)$, $\mathsf{g}_n^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1)$. Thus, the recurrence for $\widetilde{\mathsf{g}}^{\mathrm{G}}(k)$ is of the form:

$$\begin{aligned}\widetilde{\mathsf{g}}^{\mathrm{G}}(k) &= \mathsf{g}_0^{\mathrm{G}}(k) + \mathsf{g}_1^{\mathrm{G}}(k) + \mathsf{g}_n^{\mathrm{G}}(k)\\ &= \big(\mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1)\big) + \mathsf{g}_n^{\mathrm{G}}(k-1) + \big(\mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1)\big)\\ &= \widetilde{\mathsf{g}}^{\mathrm{G}}(k-1) + \mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1)\\ &= \widetilde{\mathsf{g}}^{\mathrm{G}}(k-1) + \big(\mathsf{g}_0^{\mathrm{G}}(k-2) + \mathsf{g}_1^{\mathrm{G}}(k-2)\big) + \mathsf{g}_n^{\mathrm{G}}(k-2)\\ &= \widetilde{\mathsf{g}}^{\mathrm{G}}(k-1) + \widetilde{\mathsf{g}}^{\mathrm{G}}(k-2).\end{aligned}$$

With initial conditions $\widetilde{\mathsf{g}}^{\mathrm{G}}(1)=3$, $\widetilde{\mathsf{g}}^{\mathrm{G}}(2)=5$ we deduce $\widetilde{\mathsf{g}}^{\mathrm{G}}(k)=F_{k+3}$. Therefore, we have proved the following theorem.

Theorem 2.7. *The number of the k^{th} order non-trivial compositions over the set $\mathcal{B}_n$ is*

$$\mathtt{g}^{\mathrm{G}}(k) = F_{k+3} + \mathtt{j}(k) = \begin{cases} F_{k+3} + n - 2, & 2 \nmid n \ ; \\ n+1, & 2 \mid n \ , \ k=1 \ ; \\ n+2, & 2 \mid n \ , \ k=2 \ ; \\ F_{k+3} + n - 4, & 2 \mid n \ , \ k>2 \ . \end{cases}$$

Corollary 2.8. *If $n=3$ we have obtained formula $\mathtt{g}^{\mathrm{G}}(k) = F_{k+3} + 1$ from the first section.*

Remark 2.9. The values of the function $\mathtt{g}^{\mathrm{G}}(k)$ are given in [18] as the following sequences A001611 $(n=3)$, A000045 $(n=4)$, A157726 $(n=5)$, A157725 $(n=6)$, A157729 $(n=7)$, A157727 $(n=8)$, A187107 $(n=9)$, A187179 $(n=10)$, $(k>2)$.

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