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On the metric dimension of the total graph of a graph

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Abstract: A resolving set of a graph G is a set $S \subseteq V(G)$, such that, every pair of distinct vertices of G is resolved by some vertex in S. The metric dimension of G, denoted by $\beta(G)$, is the minimum cardinality of all the resolving sets of G. Shamir Khuller et al. [10], in 1996, proved that a graph G with $\beta(G) = 2$ can have neither K_5 nor $K_{3,3}$ as its subgraph. In this paper, we obtain a forbidden subgraph, other than K_5 and $K_{3,3}$, for a graph with metric dimension two. Further, we obtain the metric dimension of the total graph of some graph families. We also establish a Nordhaus–Gaddum type inequality involving the metric dimensions of a graph and its total graph and obtain the metric dimension of the line graph of the two dimensional grid $P_m \times P_n$.
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1 Introduction

The graphs that we consider throughout this paper are simple, finite, undirected and connected. Given a graph $G = (V, E)$, a vertex $w \in V$ resolves a pair of vertices $u, v \in V$ if $d(u, w) \neq$

$d(v, w)$. A *resolving set* of G is a set of vertices $S \subseteq V$, with the property that, every pair of distinct vertices of G is resolved by some vertex in S. That is, a resolving set $S = \{w_1, w_2, \ldots, w_k\}$ is a set of vertices in G such that, each $u \in V(G)$ is identified by a k-vector of the form $r(u|S) = (d(u, w_1), d(u, w_2), \ldots, d(u, w_k))$. The vector $r(u|S)$ is called a *metric code* or S-*location* or S-code of $u \in V(G)$.

The metric dimension of G, denoted by $\beta(G)$, is the minimum cardinality among all the resolving sets of G. A resolving set with minimum cardinality is called a *metric basis*. The vertices of G in a metric basis are called *land marks*.

The concept of resolving sets was first introduced and studied by P.J. Slater [11] under the term locating set. In fact, resolving sets were studied much earlier in the context of the coin-weighing problem [3, 4, 7]. However, working independently, F. Harary and R. A. Melter [8] obtained some more results on this parameter and coined the terms resolving sets and metric dimension. Since then, these notations came into use and got widely accepted. We follow the same as well in this paper.

Since the introduction, considerable amount of work has been carried out by various authors [16, 18, 13, 6, 2, 14, 9, 17, 19] on resolving sets and metric dimension. Also, resolving sets have been used to study some real world problems such as navigation of robots, network discovery and verification.

In this paper, we obtain a forbidden subgraph, other than K_5 and $K_{3,3}$, for a graph with metric dimension two. Further, we obtain the metric dimension of the total graph of some graph families and characterize graphs whose total graphs have metric dimension two. We establish a Nordhaus–Gaddum type inequality involving the order, diameter and metric dimensions of a graph and its total graph. We also obtain the metric dimension of the total graph and line graph of the two dimensional grid $P_m \times P_n$.

2 Some known results on metric dimension

In this section, we recall some of the earlier work on metric dimension for immediate reference in the next and subsequent sections of the paper.

Theorem 2.1 (S. Khuller, B. Raghavachari and A. Rosenfeld [10])**.** *For a simple connected graph G, $\beta(G) = 1$ if and only if $G \cong P_n$.*

Theorem 2.2 (F. Harary and R. A. Melter [8])**.** *For any integer $n \geq 3$, the metric dimension of a cycle on n vertices is 2.*

Theorem 2.3 (S. Khuller, B. Raghavachari and A. Rosenfeld [10])**.** *Let $G = (V, E)$ be a graph with metric dimension 2 and let $\{a, b\} \subset V$ be a metric basis in G. The following are true:*

1. *There is a unique shortest path P between a and b.*

2. *The degrees of a and b are at most 3.*

3. *Every other node on P has degree at most 5.*

Theorem 2.4 (F. Harary and R.A.Melter [8])**.** *For any positive integer n, $\beta(G) = n - 1$ if and only if $G \cong K_n$.*

Theorem 2.5 (G. Chartrand, D. Erwin, F. Harary and P. Zhang [5])**.** *If G is a connected graph of order n, then $\beta(G) \leq n - diam(G)$.*

In view of Theorem 2.1 and Theorem 2.5, we have the following lemma.

Lemma 2.6. *For any connected graph G on n vertices which is not a path,*

$$2 \leq \beta(G) \leq n - diam(G)$$

3 A forbidden subgraph for graphs with dimension two

In the year 1996, Shamir Khuller et al. [10] proved that a graph G with $\beta(G) = 2$ can have neither K_5 nor $K_{3,3}$ as its subgraph. In this section, we obtain a forbidden subgraph as shown in Figure 1, other than K_5 and $K_{3,3}$, for graphs of dimension two in the form of the following theorem useful to prove results in the later sections of the paper.

Theorem 3.1. *If G is a graph of order n such that $\beta(G) = 2$, then G cannot have a subgraph isomorphic to the graph H of Figure 1.*

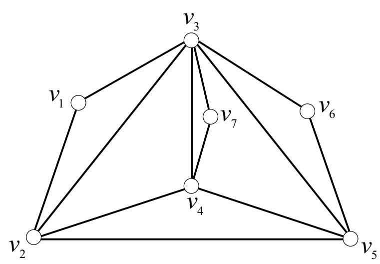

Figure 1: The graph H

Proof. Suppose that G has a subgraph isomorphic to the graph H. Let $\{v_1, v_2, \ldots, v_7\}$ be the set of vertices of the subgraph H in G. Let $S = \{u, v\}$ be a resolving set for G with minimum cardinality where $u, v \in V(G)$. Let $d(u, v_1) = a$ and $d(v, v_1) = b$. Then $r(v_1|S) = (a, b)$ and $r(v_2|S) \in \{(a, a+1), (a, b-1), (a+1, b), (a+1, b-1), (a+1, b+1), (a-1, b), (a-1, b-1), (a-1, b+1)\}$. We now analyse each of these cases as follows;

Case 1: $r(v_2|S) = (a, b+1)$

In this case, $r(v_3|S) \in \{(a+1,\ b),\ (a-1,\ b),\ (a+1,\ b+1)\}$

Subcase 1.1: Suppose $r(v_3|S) = (a+1,\ b)$.

Then, the only possible code is $r(v_4|S) = (a+1, b+1)$, but then $r(v_5|S) \in \{(a,\ b), (a,\ b+1), (a+1,\ b), (a+1,\ b+1)\}$ = $\{r(v_1|S), r(v_2|S), r(v_3|S), r(v_4|S)\}$, so that S does not resolve G.

Subcase 1.2: Suppose $r(v_3|S) = (a+1, b+1)$.

Then, $r(v_4|S) = (a+1, b)$. As discussed above, it is easy to see that S will not resolve the vertex v_5.

Subcase 1.3: Suppose $r(v_3|S) = (a-1, b)$.

Then, $r(v_4|S) = (a-1, b+1)$ which again implies that S will not resolve the vertex v_5.

Subcase 1.4: Suppose $r(v_3|S) = (a-1, b+1)$.

Then, $r(v_4|S) = (a-1, b)$ and hence v_5 cannot be resolved by S.

Case 2: $r(v_2|S) = (a, b-1)$

In this case, $r(v_3|S) \in \{(a+1, b),\ (a+1,\ b-1),\ (a-1,\ b),\ (a-1,\ b-1)\}$.

Subcase 2.1: Suppose $r(v_3|S) = (a+1,\ b)$.

Then, $r(v_4|S) = (a+1,\ b-1)$ so that S will not resolve v_5.

Subcase 2.2: Suppose $r(v_3|S) = (a+1,\ b-1)$.

Then, $r(v_4|S) = (a+1,\ b)$ which implies that S will not resolve v_5.

Subcase 2.3: Suppose $r(v_3|S) = (a-1,\ b)$.

Then again, S will not resolve v_5 as $r(v_4|S) = (a-1,\ b+1)$.

Subcase 2.4: Suppose $r(v_3|S) = (a-1,\ b-1)$.

Then, $r(v_4|S) = (a-1,\ b)$ and hence again v_5 is unresolved by S.

Case 3: $r(v_2|S) = (a+1,\ b)$

In this case, $r(v_3|S) \in \{(a,\ b+1), (a,\ b-1), (a+1,\ b-1), (a+1,\ b+1)\}$.

Subcase 3.1: Suppose $r(v_3|S) = (a,\ b+1)$.

In this case, $r(v_4|S) = (a+1,\ b+1)$, so again S will not resolve v_5.

Subcase 3.2: Suppose $r(v_3|S) = (a+1,\ b-1)$.

In this case, $r(v_4|S) = (a+1,\ b)$, so S will not resolve v_5.

Subcase 3.3: Suppose $r(v_3|S) = (a+1,\ b-1)$.

In this case, $r(v_4|S) = (a,\ b-1)$ so that metric code of v_5 will repeat.

Subcase 3.4: Suppose $r(v_3|S) = (a+1,\ b+1)$.

In this case, $r(v_4|S) = (a,\ b+1)$ and hence again v_5 is unresolved.

Case 4: $r(v_2|S) = (a+1,\ b-1)$.

In this case, $r(v_3|S) \in \{(a,\ b-1), (a+1,\ b)\}$.

Subcase 4.1: Suppose $r(v_3|S) = (a,\ b-1)$.

In this case, $r(v_4|S) = (a+1,\ b)$, so again S will not resolve v_5.

Subcase 4.2: Suppose $r(v_3|S) = (a+1,\ b)$.

In this case, $r(v_4|S) = (a,\ b-1)$, so S will not resolve v_5.

Case 5: $r(v_2|S) = (a+1,\ b+1)$.

In this case, $r(v_3|S) \in \{(a,\ b+1), (a+1,\ b)\}$.

Subcase 5.1: Suppose $r(v_3|S) = (a,\ b+1)$.
In this case, $r(v_4|S) = (a+1,\ b)$, so again S will not resolve v_5.

Subcase 5.2: Suppose $r(v_3|S) = (a+1,\ b)$.
In this case, $r(v_4|S) = (a,\ b+1)$, so S will not resolve v_5.

Case 6: $r(v_2|S) = (a-1,\ b)$.
In this case, $r(v_3|S) \in \{(a,\ b+1), (a,\ b-1), (a-1,\ b+1), (a-1,\ b-1)\}$.

Subcase 6.1: Suppose $r(v_3|S) = (a,\ b+1)$.
In this case, $r(v_4|S) = (a-1,\ b+1)$, so again S will not resolve v_5.

Subcase 6.2: Suppose $r(v_3|S) = (a,\ b-1)$.
In this case, $r(v_4|S) = (a-1,\ b-1)$, so S will not resolve v_5.

Subcase 6.3: Suppose $r(v_3|S) = (a-1,\ b+1)$.
In this case, $r(v_4|S) \in \{(a,\ b+1), (a-2,\ b), (a-2,\ b+1)\}$, so again S will not resolve v_5.

Subsubcase 6.3.1: $r(v_4|S) = (a,\ b+1)$.
In this case, S will not resolve v_5.

Subsubcase 6.3.2 $r(v_4|S) = (a-2,\ b)$.
In this case, $r(v_5|S) = (a-2,\ b+1)$ and hence S will not resolve V_6.

Subsubcase 6.3.3 $r(v_4|S) = (a-2,\ b+1)$.
In this case, $r(v_5|S) = (a-2,\ b)$ and hence S will not resolve V_6.

Case 7: $r(v_2|S) = (a-1,\ b-1)$.
In this case, $r(v_3|S) \in \{(a,\ b-1), (a-1,\ b)$.

Subcase 7.1: Suppose $r(v_3|S) = (a,\ b-1)$.
In this case, $r(v_4|S) \in \{(a,\ b-2), (a-1,\ b), (a-1,\ b-2)$.

Subsubcase 7.1.1: $r(v_4|S) = (a,\ b-2)$.
In this case, $r(v_5|S) = (a-1,\ b-2)$ and S will not resolve v_6.

Subsubcase 7.1.2: $r(v_4|S) = (a-1,\ b)$.
In this case, S will not resolve v_5.

Subsubcase 7.1.3: $r(v_4|S) = (a-1,\ b-2)$.
In this case, $r(v_5|S) = (a,\ b-2)$ and $v_6 \in \{(a+1,\ b-1), (a+1,\ b-2)\}$.

Subsubsubcase 7.1.3.1: $r(v_6|S) = (a+1,\ b-1)$.
In this case, S will not resolve the vertex v_7.

Subsubsubcase 7.1.3.2: $r(v_6|S) = (a+1,\ b-2)$.
In this case, S will not resolve the vertex v_7.

Case 8: $r(v_2|S) = (a-1,\ b+1)$.
In this case, $r(v_3|S) \in \{(a,\ b+1), (a-1,\ b)\}$.

Subcase 8.1: Suppose $r(v_3|S) = (a,\ b+1)$.
In this case, $r(v_4|S) = (a-1,\ b)$, so S will not resolve v_5.

Subcase 8.2: $r(v_3|S) = (a-1,\ b)$.

In this case, $r(v_4|S) = (a,\ b+1)$, so S will not resolve v_5.

As discussed above, in each of the cases, at least two vertices of H receive the same metric code with respect to the set of vertices $S = \{u, v\}$ so that S cannot be a resolving set for G. Hence, $\beta(G) > 2$, a contradiction. $\square$

The above theorem is central to this paper and will be used to prove other results in the later sections.

4 Dimension of total graph of some classes of graphs

The *total graph* of a graph G, denoted by $T(G)$, is defined as the graph with vertex set $V(G) \cup E(G)$, such that, two vertices x and y in $T(G)$ are adjacent if and only if x and y are either adjacent or incident in G.

The following are some observations on the total graph of a graph.

Observation 4.1. *If G_1 and G_2 are two graphs such that $T(G_1) \cong T(G_2)$, then $G_1 \cong G_2$.*

Observation 4.2. *From the definition of total graph, for any non-trivial graph G, it is clear that $T(G) \ncong P_n$ for any $n \in Z^+$.*

Observation 4.3. *Since $T(G) \ncong P_n$ if $n > 1$, in view of Theorem 2.1, it follows that $\beta(T(G)) \geq 2$ whenever order of G is at least two.*

Observation 4.4. *If G is a graph of order p and size q, then the order of the total graph $T(G)$ is $p + q$.*

In this section, we obtain the metric dimension of the total graph of three standard classes of graphs namely cycle, path and star graph. Also, we prove a Nordhaus–Gaddum type result [12] bounding the sum of the metric dimensions of a graph and its total graph in terms of its order and diameter.

Theorem 4.5. *For $n \geq 3$, the metric dimension of the total graph of a cycle graph C_n is equal to 3.*

Proof. Since $T(C_n)$ is a 4-regular graph, it has no vertex of degree 3 and hence by condition 2 of Theorem 2.3, it follows that $T(G)$ cannot have a metric basis of cardinality 2 so that $\beta(T(G)) \geq 3$.

The codes generated in Figure 2, Figure 3 and Figure 4, show that $\beta(T(C_n)) = 3$ for $n = 3, 4, 5$.

We now consider the case $n \geq 6$. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_i : e_i = v_i v_{i+1}, 1 \leq i \leq n-1\} \cup \{e_n = v_n v_1\}$. Consider a subset $S = \{v_1, e_1, v_2\}$ of vertices of $T(C_n)$. Now, the codes for the vertices of $T(C_n)$ generated with respect to S are:

1. $r(e_i|S) = (i, i-1, i-1)$, for $2 \leq i \leq \lceil \frac{n}{2} \rceil$

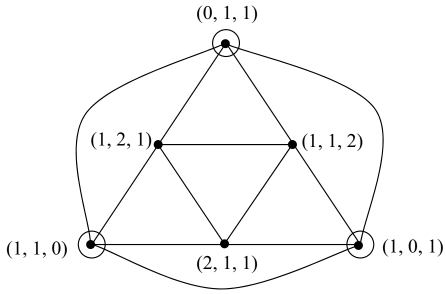

Figure 2: The graph $T(C_3)$ and its metric basis

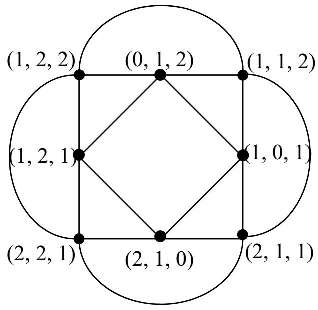

Figure 3: The graph $T(C_4)$ and its metric basis

2. $r(v_i|S) = (i-1, i-1, i-2)$, for $\begin{cases} 3 \le i \le \frac{n+1}{2} & \text{if } n \text{ is odd} \\ 3 \le i \le \frac{n+2}{2} & \text{if } n \text{ is even} \end{cases}$

3. $r(e_{n-i}|S) = (i+1, i+1, i+2)$, for $\begin{cases} 0 \le i \le \frac{n-1}{2} & \text{if } n \text{ is odd} \\ 0 \le i \le \frac{n-4}{2} & \text{if } n \text{ is even} \end{cases}$

4. $r(v_{n-i}|S) = (i, i+1, i+1)$, for $\begin{cases} 1 \le i \le \frac{n}{2} & \text{if } n \text{ is even} \\ 1 \le i \le \frac{n-3}{2} & \text{if } n \text{ is odd} \end{cases}$

5. $r(v_{\frac{n+3}{2}}|S) = \left(\frac{n-1}{2}, \frac{n+1}{2}, \frac{n-1}{2}\right)$, only when n is odd.

It is easy to verify that the codes of all the vertices of C_n are distinct so that S is a resolving set for $T(C_n)$. Thus $\beta(T(C_n)) = 3$. □

Theorem 4.6. *For a graph G, $\beta(T(G)) = 2$ if and only if G is a path P_n , $n \ge 2$.*

Proof. Consider the graph $G = P_n$, a path on n vertices. By Observation 4.3, it suffices to prove that $\beta(T(P_n)) \le 2$. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of P_n with v_i adjacent to v_{i+1} for all $i = 1,\ 2,\ \ldots, n-1$. Let $e_i = v_iv_{i+1}$, for $i = 1, 2, \ldots\ldots, n-1$. The set $V(T(P_n)) = \{v_1,\ v_2,\ \ldots, v_n\} \cup \{e_1,\ e_2,\ \ldots,\ e_{n-1}\}$.

Define $S = \{v_1, v_n\}$. The vertices v_i of $T(P_n)$ for $1 < i \le n$, and the vertices e_j for $1 \le j \le n-1$ are uniquely identified by S as $r(v_i|S) = (i-1,\ n-i)$; $r(e_i|S) = (i,\ n-i)$. Thus S serves as a resolving set for $T((P_n))$ and hence $\beta(T(P_n)) \le |S| = 2$.

Conversely, let G be a graph of order n and $\beta(T(G)) = 2$. Suppose to contrary that G is not a path. Then G has at least one vertex of degree three or every vertex of degree two.

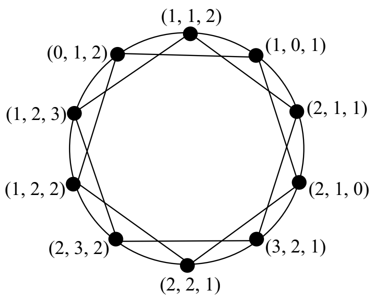

Figure 4: The graph $T(C_5)$ and its metric basis

Case 1: G has at least one vertex of degree three.

In this case, $T(G)$ has a subgraph isomorphic to the graph H shown in Figure 1. Hence by Theorem 3.1, we get $\beta(T(P_n)) > 2$, a contradiction.

Case 2: G is 2-regular.

In this case, G is a cycle and hence by Theorem 4.5, it follows that $\beta(T(G)) = 3 > 2$, a contradiction.

From either of the cases, it follows that G is a path. □

Theorem 4.7. *For any integer* $n \geq 1$,

$$\beta(T(K_{1,n})) = \begin{cases} 2, & \textit{if} \;\; n = 1, \\ n, & \textit{if} \;\; 2 \leq n \leq 4, \\ n-1, & \textit{if} \;\; n \geq 5. \end{cases}$$

Proof. Consider the graph $K_{1,n}$ with v_0 as the central vertex and $v_1, v_2, \ldots, v_n$ as the pendant vertices. Let the edges be $e_i = v_o v_i$ for $1 \leq i \leq n$.

For $n = 1$ and $n = 2$, the result follows from Theorem 4.6. For $n \geq 3$, we consider the following cases.

Case 1: $n = 3$.

In this case, as $K_{1,3} \ncong P_3$, it follows that $\beta(T(K_{1,3})) \geq 3$. On the other hand, the reverse inequality $\beta(T(K_{1,3})) \leq 3$ follows from Figure 5.

Case 2: $n = 4$.

In this case, the code generated in graph shown in Figure 6 implies that $\beta(T(K_{1,4})) \leq 4$. To prove the reverse inequality, we now show that $|S| \geq 4$ for every resolvable set S. If possible, let S be a metric basis of $T(K_{1,4})$ with cardinality three. Then, we have the following;

Subcase 1: S contains only edges of $K_{1,4}$.

Without loss of generality, we take $S = \{e_1, e_2, e_3\}$. Then for the vertices $v_0, e_4 \in V(T(K_{1,4})) - S$, we get $r(v_0|S) = r(e_4|S) = (1, 1, 1)$, a contradiction.

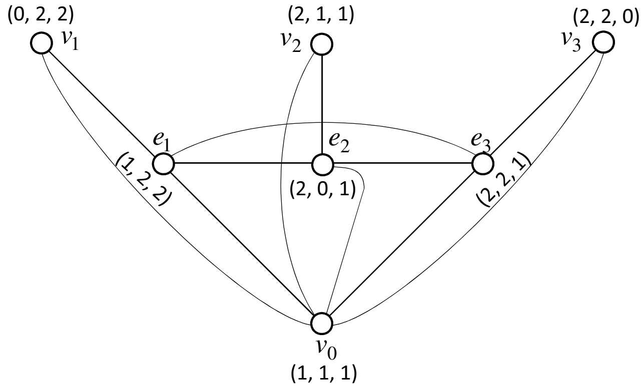

Figure 5: A metric code for the graph $T(K_{1,3})$.

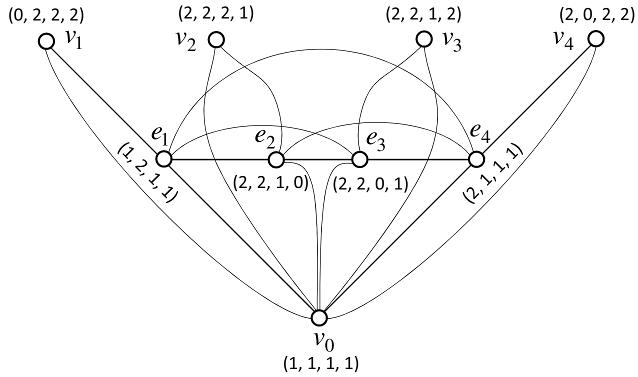

Figure 6: Metric codes of total graph of a star $K_{1,4}$.

Subcase 2: S contains two edges and one vertex of $K_{1,4}$.

In this case the vertex in S should be non-adjacent to the edges in S, else the two (non-central) vertices in $V(T(K_{1,4})) - S$ receive the same code (2,2,2) which is not possible. Therefore, without loss of generality we assume $S = \{e_1, e_2, v_3\}$. But then $r(e_3|S) = (1,1,1) = r(v_0|S) = (1,1,1)$, a contradiction.

Subcase 3: S contains one edge and two vertices of $K_{1,4}$.

In this case none of the vertices in S is adjacent to the edge in S, else $V(T(K_{1,4}) - S$ has two vertices that receive the same code (2,2,2) which is not possible. Therefore, without loss of generality, we take $S = \{e_1, v_2, v_3\}$. But then, $r(v_1|S) = (1,2,2) = r(e_4|S)$, a contradiction.

Subcase 4: S contains only three vertices of $K_{1,4}$.

Without loss of generality, we take $S = \{v_1, v_2, v_3\}$. Then $r(v_4|S) = (2,2,2) = r(e_4|S)$, a contradiction.

Hence S should contain at least four elements so that $\beta(T(K_{1,4})) \geq 4$. Hence $\beta(T(K_{1,4})) = 4$.

Case 3: $n \geq 5$.

Necessity: Let S be a minimal resolving set of $T(K_{1,n})$. Due to minimality, it then follows that the central vertex v_0 is not in S as v_0 is the only vertex adjacent to every vertex in $T(K_{1,n})$. We now show that $|S| \geq n-1$.

Otherwise, $|S| \leq n-2$. Then by pigeonhole principle there exist at least two vertices in $T(K_{1,n})$ not in S, corresponding to a vertex of $K_{1,n}$. Without loss of generality, let these two vertices be v_1, v_2. But then, at least one of the vertices e_1, e_2 of $T(K_{1,n})$ should be in S, otherwise which both v_1, v_2 are at an equal distance, that being two, in $T(K_{1,n})$ from every element in S. We choose $e_2 \in S$ due to symmetry. Then, there exists one more vertex $v_3 \notin S$ which corresponds to a vertex of $K_{1,n}$ since $|S| \leq n-2$. Now, following the same argument, for the new pair (v_1, v_3), S should contain at least one of the edges e_1, e_3. Again, because of symmetry, we take $e_3 \in S$. Continuing this argument, we end up with the set S containing only edges of $K_{1,n}$. Finally, since $|S| \leq n-2$ and contains only edges of $K_{1,n}$, it follows that $v_1, e_n \notin S$. Now for any $x \in S$, $d(x, v_1) = d(x, v_n) = 2$ (since in $d(v_i, x) = 1$ in $T(K_{1,n})$ if and only if $x = v_0$ or e_i), a contradiction to the fact that S is a resolving set of $T(K_{1,n})$. Therefore, $\beta(T(K_{1,n})) \geq n-1$.

Sufficiency: Consider the set $S = \begin{cases} \{v_1, e_2, v_3, e_4, \ldots, e_{n-2}, v_{n-1}\} & \text{if } n \text{ is even} \\ \{v_1, e_2, v_3, e_4, \ldots, v_{n-2}, e_{n-1}\} & \text{if } n \text{ is odd} \end{cases}$,

Then, as $d(v_i, e_j) = 2$ for $i \neq j$ and $d(e_i, e_j) = 1$ for all $i \neq j$, it easily follows that

For each $e_i \in V(T(K_{1,n})) - S$,

$r(e_i|S) = (a_1, a_2, \ldots, a_{n-1})$, where $a_k = \begin{cases} 1 & \text{if } k = i \\ 2 & \text{otherwise} \end{cases}$

For each $v_i \in V(T(K_{1,n})) - S$,

$r(v_i|S) = (a_1, a_2, \ldots, a_{n-1})$, where $a_k = \begin{cases} 1 & \text{if } k = i \text{ or } i = 0 \\ 2 & \text{otherwise} \end{cases}$

This shows that S is a resolving set. Thus $\beta(T(K_{1,n})) \leq n-1$.

Hence the theorem. □

We end the section with a result bounding the sum of the metric dimensions of a graph and its total graph in terms of its order and diameter, which follows from the previous results.

Theorem 4.8. *If G is a connected graph of order at least $n \geq 2$ and diameter d, then $3 \leq \beta(G) + \beta(T(G)) \leq 2n - d$ and $\beta(G) + \beta(T(G)) = 3$ if and only if $G = P_n$.*

5 Dimension of the total graph and line graph of two dimensional grid

If $G = (V, E)$ is a graph with $|E| \geq 1$, the *line graph* of G, denoted $L(G) = (V_1, E_1)$, is a graph with $V_1 = E(G)$ with the property that two vertices in V_1 are adjacent if and only if the corresponding edges in G are adjacent.

The two dimensional grid is the graph obtained by taking the cartesian product of the paths P_m and P_n and is denoted $P_m \times P_n$. In this section, we obtain the metric dimension of the total graph and the line graph of the two dimensional grid $P_m \times P_n$.

Theorem 5.1. *The metric dimension of total graph of a 2-dimensional grid is three.*

Proof. Let $G = P_m \times P_n$. The case $m = 2,\ n = 2$ follows directly by Theorem 4.5 as $P_2 \times P_2 = C_4$. For the other cases, due to symmetry of cartesian product, it suffices to consider only the case $m > 2$ and $n \geq 2$. In this case, the graph $T(G)$ contains a subgraph isomorphic to the graph H of Figure 1, so by Lemma 3.1, it follows that $\beta(T(G)) \geq 3$.

Now, to prove the reverse inequality, let $\{u_1, u_2, \dots, u_m\}$ and $\{v_1, v_2, \dots, v_n\}$ be the set of vertices of the graph P_m and P_n respectively. Classify the vertices of $T(P_m \times P_n)$ as;

Vertices of $P_m \times P_n$: $v_{i,j} = (u_i, v_j)$, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Edges of $P_m \times P_n$: **Type 1** : $R_{i,j} = v_{i,j}v_{i,j+1}$ for $1 \leq i \leq m; 1 \leq j \leq n-1$ and
Type 2 : $C_{i,j} = v_{i,j}v_{i+1,j}$ for $1 \leq i \leq m-1; 1 \leq j \leq n$.

So, the vertex set of $T(G)$ is $\{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \bigcup \{R_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n-1\} \bigcup \{C_{i,j} : 1 \leq i \leq m-1, 1 \leq j \leq n\}$.

Now, consider the subset $S = \{v_{1,1}, v_{1,n}, v_{m,n}\}$ of vertex set of G. It is easy to see that the S-codes for each vertices of G are as follows;

$$r(v_{i,j}|S) = (d(v_{i,j}, v_{1,1}), d(v_{i,j}, v_{1,n}), d(v_{i,j}, v_{m,n})) = (i+j-2, n+i-j-1, m+n-i-j)$$

$$r(R_{i,j}|S) = (d(R_{i,j}, v_{1,1}), d(R_{i,j}, v_{1,n}), d(R_{i,j}, v_{m,n})) = (i+j-1, n+i-j-1, m+n-i-j)$$

$$r(c_{i,j}|S) = (d(c_{i,j}, v_{1,1}), d(c_{i,j}, v_{1,n}), d(c_{i,j}, v_{m,n})) = (i+j-1, n+j-i, m+n-i-j)$$

Claim: S is a resolving set.

If not, then there exists two distinct vertices u and v in $T(G)$ such that $r(u|S) = r(v|S)$. Now, as u and v are interchangeable, we have only the following cases;

Case 1: $u = v_{i,j}$ and $v = v_{k,l}$
In this case, $r(u|S) = r(v|S) \Rightarrow (i+j-2, n+i-j-1, m+n-i-j) = (k+l-2, n+k-l-1, m+n-k-l)$, This yields $i = k$ and $j = l$, so $u = v$, a contradiction.

Case 2: $u = R_{i,j}$ and $v = R_{k,l}$
In this case, $r(u|S) = r(v|S) \Rightarrow (i+j-1, n+i-j-1, m+n-i-j) = (k+l-1, n+k-l-1, m+n-k-l)$ This yields to $i = k$ and $j = l$, so $u = v$ a contradiction.

Case 3: $u = C_{i,j}$ and $v = C_{k,l}$
In this case, $r(u|S) = r(v|S) \Rightarrow (i+j-1, n+i-j-1, m+n-i-j) = (k+l-1, n+k-l-1, m+n-k-l)$ This yields $i = k$ and $j = l$, so $u = v$ a contradiction.

In the remaining cases, that is, $u = v_{i,j}$ and $v = R_{k,l}$, $u = v_{i,j}$ and $v = C_{k,l}$ and $u = R_{i,j}$ and $v = C_{k,l}$, we get contradictory expressions as well.

Hence the claim. Thus, S is a resolving set so that $\beta(T(G)) \leq 3$.
To conclude, $\beta(T(G)) = 3$. $\square$

Theorem 5.2. *For integers $m \geq 2$, $n \geq 2$, with $m \geq n$,*

$$\beta(L(P_m \times P_n)) = \begin{cases} 2, & if \quad m \geq 2 \quad and \quad n = 2, \\ 3, & if \quad n \geq 3 \quad and \quad m \geq 3 \end{cases}$$

Proof. Let $u_1, u_2, \ldots, u_m$ be the vertices of P_m such that u_i is adjacent to u_{i+1} for $i = 1, 2, \ldots, m-1$. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n such that v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$. Then the vertex (u_i, v_j) is adjacent to (u_l, v_k) in $P_m \times P_n$ if and only if either [$l = i \pm 1$ and $j = k$] or [$k = j \pm 1$ and $i = l$].

Let us categorise the edges of $P_m \times P_n$ as row edges $r_{i,j} = \{(u_i, v_j), (u_i, v_{j+1})\}$ for $1 \leq i \leq m$, $1 \leq j \leq n-1$ and column edges $c_{i,j} = \{(u_i, v_j), (u_{i+1}, v_j)\}$ for $1 \leq i \leq m-1$, $1 \leq j \leq n$.

For the case $m = 2, n = 2$, the graph $P_2 \times P_2 \cong C_4$ and $L(C_n) = C_n$, so that, by Theorem 2.2 $\beta((L(P_2 \times P_2)) = \beta(L(C_4)) = \beta(C_4) = 2$.

For $m > 2$, $n = 2$, as $P_m \times P_n$ is not a path, it is obvious by Lemma 2.6 that $\beta(L(P_m \times P_2)) \geq 2$.

Now to prove the reverse inequality, let $S = \{c_{1,1}, c_{m-1,1}\}$. Then the codes for the vertices of the line graph of $P_m \times P_2$ with respect S are

$$\begin{array}{lcll}
r(c_{1,j}|S) & = & (j-1, m-j-1) & \text{for } 1 < j < m-1 \\
r(c_{2,j}|S) & = & (j, m-j), & \text{for } 1 < j < m-2 \\
r(r_{i,1}|S) & = & (i-1, m-i), & \text{for } 1 < i < m \\
r(c_{1,2}|S) & = & (2, m-1), & \\
r(c_{m-1,2}|S) & = & (m-1, 2), & \\
r(r_{1,1}|S) & = & (1, m-1), & \\
r(r_{m,1}|S) & = & (m-1, 1) &
\end{array}$$

It is easy to verify that no two vertices in $P_m \times P_2$ receive the same S-code. Thus, $\beta(L(P_m \times P_2)) = 2$.

We now consider the case $m > 3, n \geq 3$.

Claim 1: $\beta(L(P_m \times P_n)) \geq 3$.

If not, then there exists a resolving set $S = \{u, v\}$ for $L(P_m \times P_n)$. But then, as the line graph of $P_m \times P_n$ contains exactly eight vertices namely $r_{1,1}$, $c_{m-1,1}$, $c_{1,n}$, $c_{m-1,n}$, $r_{1,n-1}$, $r_{m,n-1}$, $r_{1,1}$, $r_{m,1}$ of degree three, by Theorem 2.3, it follows that both u and v must be any two of these eight vertices of degree three. Also, by the uniqueness of shortest path between u and v, again by Theorem 2.3, the only possibilities are the following.

Case 1: $u = r_{1,1}, v = c_{1,1}$

In this case we get $r(r_{2,1}|S) = r(c_{2,1}|S)$, a contradiction to the fact that S is a resolving set.

Case 2: $u = r_{1,1}, v = r_{1,m-1}$

In this case we get $r(r_{2,m-1}|S) = r(c_{m-1,2}|S)$, again a contradiction.

Case 3: $u = r_{1,1}, v = c_{m-1,1}$

In this case we get $r(r_{m-2,2}|S) = r(c_{m-1,1}|S)$, a contradiction. Hence the claim.

The other cases follow by symmetry.

Claim 2: $\beta(P_m \times P_n) \leq 3$.

Let $S = \{c_{1,1}, c_{1,m-1}, c_{n,2}\}$. Then

$$\begin{aligned} r(c_{i,j}|S) &= (d(c_{i,j}, r_{1,1}), d(c_{i,j}, c_{1,m-1}), d(c_{i,j}, c_{n,2})) \\ r(r_{i,j}|S) &= (d(r_{i,j}, r_{1,1}), d(r_{i,j}, c_{1,m-1}), d(r_{i,j}, c_{n,2})) \end{aligned}$$

implies that

$$r(c_{i,j}|S) = \begin{cases} (i+j-2, m-j+i-2, n-i+j-2) & \text{if} \quad 2 < j < m-1, 1 \leq i \leq n-1 \\ (i+j-1, m+j-i-2, n+i-j) & \text{if} \quad j = 1, 1 \leq i \leq n \\ (i+j-2, m-j+i-2, n-i+j-1) & \text{for} \quad j = 2, 1 \leq i \leq n \\ (i+j-2, m-j+i-1, n-i+j-2) & \text{for} \quad j = m-1, 1 \leq i \leq n \end{cases}$$

$$r(r_{i,j}|S) = \begin{cases} (i+j-2, m+j-i-1, n+i-j-3) & \text{if } 2 < j < m-1, 1 \leq i \leq n-1 \\ (i+j-1, m-j+i-1, n-i+j) & \text{if } j = 1, 1 \leq i \leq n \\ (i+j-2, m-j+i-1, n-i+j-2) & \text{for } j = 2, 1 \leq i \leq n \\ (i+j-2, m+j-i-1, n+i-j-3) & \text{for } j = m-1, 1 \leq i \leq n \\ (i+j-2, m-j+i, n-i+j-3) & \text{for } i = m, 1 \leq j \leq n-1 \end{cases}$$

From the above computation, it is easy to observe that no two vertices receive the same S-code. Therefore S is a resolving set with cardinality three. Hence the claim.

From Claim 1 and Claim 2, it is clear that $\beta(L(P_m \times P_n)) = 3$. □

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