Identities for balancing numbers using generating function and some new congruence relations

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Abstract: It is well-known that the balancing numbers are the square roots of the triangular numbers and are the solutions of the Diophantine equation $1 + 2 + \ldots + (n - 1) = (n + 1) + (n + 2) + \ldots + (n + r)$, where $r$ is the balancer corresponding to the balancing number $n$. Thus if $n$ is a balancing number, then $8n^2 + 1$ is a perfect square and its positive square root is called a Lucas-balancing number. The goal of this paper is to establish some new identities of these numbers.

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1 Introduction

A. Behera et.al [2] introduced the sequence of balancing numbers as follows. A positive integer $n$ is called a balancing number with balancer $r$ if it is the solution of Diophantine equation

$$1 + 2 + \ldots + (n - 1) = (n + 1) + (n + 2) + \ldots + (n + r).$$

The balancing numbers though obtained from a simple Diophantine equation, are very useful for the computation of square triangular numbers. An important result about balancing numbers is
that, $B$ is balancing number if and only if $B^2$ is a triangular number i.e. $8B^2+1$ is a perfect square. For each balancing number $B$, $C = \sqrt{8B^2 + 1}$ is called a Lucas-balancing number [9, 10]. First four balancing numbers are 1, 6, 35 and 204 with balancers 0, 2, 14 and 84 respectively. Let $B_n$ and $C_n$ are respectively denoted by $n^{th}$ balancing number and $n^{th}$ Lucas-balancing number. The balancing numbers satisfy the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}; \quad n \geq 2,$$

with $B_1 = 1$ and $B_2 = 6$ [2] whereas, the Lucas-balancing numbers defined recursively by

$$C_{n+1} = 6C_n - C_{n-1}; \quad n \geq 2,$$

with $C_1 = 3$ and $C_2 = 17$ [9]. Many useful identities involving balancing and Lucas-balancing numbers are available in the literature. One can go through [1, 3, 4, 5, 6, 7, 8, 11, 12]. There are many well-known relationships between balancing and Lucas-balancing numbers. Most of the relationships were established from the Binet’s formulas

$$B_n = \frac{\lambda^n - \lambda^{-n}}{2\sqrt{8}}, \quad C_n = \frac{\lambda^n + \lambda^{-n}}{2},$$

where $\lambda = 3 + \sqrt{8}$ and $\lambda^{-1} = 3 - \sqrt{8}$. It is well known that matrices are used to represent the Fibonacci numbers. Also these can be used to represent balancing numbers and their related sequences. In [13], Ray has introduced a second order balancing $Q_B$ matrix whose entries are the first three balancing numbers 0, 1 and 6. He has also shown that the $n^{th}$ power of the balancing matrix $Q_B$ is given by

$$Q_B^n = \begin{pmatrix}
B_{n+1} & -B_n \\
B_n & -B_{n-1}
\end{pmatrix}.$$

This matrix representation turns out to be an elegant way of finding relationships between the balancing and Lucas-balancing numbers. Ray, in [16], has established some balancing and Lucas-balancing sums using matrix method. The observation $\det(Q_B) = 1 = \det(Q_B^n)$ at once gives the Cassini formula for the balancing numbers $B_k^2 - B_{k+1}B_{k-1} = 1$.

In this article, we establish some combinatorial properties of balancing numbers and then establish some new congruences relations for these numbers.

## 2 Some known properties for balancing numbers by matrix method

In this section, we recover some well known properties of balancing numbers by matrix method.

### 2.1 Binet’s formula

Consider a pair of two consecutive vectors $(B_{n+1}, B_n)$ from the balancing sequence

$$\ldots, B_n, B_{n+1}, B_{n+2}, B_{n+3}, \ldots$$
In fact, the recurrence relation can be described in the following manner

\[
\begin{pmatrix}
6 & -1 \\
\frac{B_{n+1}}{B_n}
\end{pmatrix}
= 6B_{n+1} - B_n = B_{n+2}.
\]

The two consecutive vectors can be written in a matrix form as

\[
\begin{pmatrix}
B_{n+2} & -B_{n+1} \\
B_{n+1} & -B_n
\end{pmatrix},
\]

so that for \( n = 0 \) we recover the balancing matrix \( Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \).

Also, it can be observed that

\[
Q_B^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} B_{n+1} \\ B_n \end{pmatrix}.
\]

The power of a non-diagonal matrix is difficult to compute, therefore in order to use (4) to describe \( B_n \), we must diagonalize \( Q_B \) as follows: Clearly \( \lambda_1 = 3 + \sqrt{8} \) and \( \lambda_2 = 3 - \sqrt{8} \) are eigenvalues of the matrix \( Q_B \). These roots satisfy the following relations

\[
\lambda_1^2 = 6\lambda_1 - 1, \quad \lambda_2^2 = 6\lambda_2 - 1, \\
\lambda_1\lambda_2 = 1, \quad \lambda_1 + \lambda_2 = 6, \quad \lambda_1 - \lambda_2 = 2\sqrt{8}.
\]

Also it has been observed that the eigenvectors corresponding to the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are of the form \( (\lambda_1, 1)^T \) and \( (\lambda_2, 1)^T \) respectively. Putting these eigenvectors into a change of basis matrix \( P \) which is in the form

\[
P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}.
\]

If \( \widetilde{Q}_B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \) denotes the diagonalization of the balancing matrix \( Q_B \) and since \( \lambda_1 - \lambda_2 = 2\sqrt{8} \), we have

\[
Q_B = P\widetilde{Q}_BP^{-1},
\]

where \( P^{-1} = \frac{1}{2\sqrt{8}} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \). It follows that

\[
Q_B^n = P\widetilde{Q}_B^n P^{-1}.
\]

Thus by (4), we have

\[
\begin{pmatrix} B_{n+1} \\ B_n \end{pmatrix} = \frac{1}{2\sqrt{8}} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
= \frac{1}{2\sqrt{8}} \begin{pmatrix} \lambda_1^{n+1} + \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}.
\]

43
Comparing the second row and first column element, we obtain
\[ B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \]
which is popularly known as Binet’s formula for balancing numbers.

### 2.2 Generating function of balancing numbers by matrix method

Recall that the generating function \( G(x) \) for any sequence \( \{a_n\} \) is given by the formula
\[ G(x) = \sum_{k=0}^{\infty} a_k x^k. \]

Behera et.al established the generating function for balancing numbers in [2] as
\[ g(s) = \frac{s}{1 - 6s + s^2}. \]

This result can also be obtained by matrix method. Consider the matrix
\[ I - sQ_B = \begin{pmatrix} 1 - 6s & s \\ -s & 1 \end{pmatrix}, \quad (5) \]
where \( I \) be the identity matrix same order as \( Q_B \). Since the determinant value of the matrix (5) is \( 1 - 6s + s^2 \), its inverse will be
\[ (I - sQ_B)^{-1} = \frac{1}{1 - 6s + s^2} \begin{pmatrix} 1 & -s \\ s & 1 - 6s \end{pmatrix}. \quad (6) \]

Therefore, if \( g(s) = \sum_{k=0}^{\infty} s^k Q_B^k = (I - sQ_B)^{-1} \) be the generating function, then using (6) we can write
\[ s^0 Q_B^0 + s^1 Q_B^1 + s^2 Q_B^2 + \ldots = \begin{pmatrix} 1 & -s \\ s & 1 - 6s \end{pmatrix} \begin{pmatrix} 1 - 6s + s^2 \\ -s \end{pmatrix} = g(s) = \frac{s}{1 - 6s + s^2}. \quad (7) \]

If we choose \( Q_B^k \) instead of \( Q_B \) and proceeding as before, we get
\[ \det(I - sQ_B^k) = 1 - s(B_{k+1} - B_{k-1}) + s^2. \]

Since \( B_{k+1} - B_{k-1} = 2C_k \) where \( C_k \) is the \( k^{th} \) Lucas-balancing number, it follows that
\[ \det(I - sQ_B^k) = 1 - 2sC_k + s^2, \]
and therefore its inverse is given by
\[ (I - sQ_B^k)^{-1} = \frac{1}{1 - 2sC_k + s^2} \begin{pmatrix} 1 + sB_{k-1} & -sB_k \\ sB_k & 1 - sB_{k+1} \end{pmatrix}. \]
Since \( \sum_{n=0}^{\infty} s^n Q_{nk}^k = (I - s Q_{nk}^k)^{-1} \), it follows that

\[
B_{0k} + B_{1k} + sB_{2k} + s^2B_{3k} + \ldots = \frac{B_k}{1 - 2sC_k + s^2},
\]

for instance, with \( k = 2 \) we have

\[
B_0 + B_2 + sB_4 + s^2B_6 + \ldots = \frac{6}{1 - 34s + s^2}.
\]

### 3 Some combinatorial identities using generating function

In this section, we will establish some new results involving balancing numbers with the help of generating function.

**Theorem 3.1.** If \( B_n \) denotes the \( n^{th} \) balancing number, then

\[
B_n = \left\lfloor \frac{n + 1}{2} \right\rfloor \sum_{i=0}^{\left\lfloor \frac{n - 1}{2} \right\rfloor} (-1)^i \binom{n - i - 1}{i} 6^{n-2i-1}.
\]

**Proof.** By virtue of (7), we get

\[
B_0 + sB_1 + s^2B_2 + \ldots = s \left( 1 - (6s - s^2) \right)^{-1} = s \left[ 1 + (6s - s^2) + (6s - s^2)^2 + \ldots \right].
\]

Equating the coefficient of \( s^n \) from both the sides, we obtain

\[
B_n = (6s - s^2)^{n-1}.
\]

Expanding the right hand side expression of the above equation binomially we obtain the desired result.

The proof of the following result is analogous to Theorem 3.1

**Theorem 3.2.** If \( B_k \) and \( C_k \) are the \( k^{th} \) balancing and \( k^{th} \) Lucas-balancing numbers respectively then

\[
B_{nk} = B_k \left\lfloor \frac{n + 1}{2} \right\rfloor \sum_{i=0}^{\left\lfloor \frac{n - 1}{2} \right\rfloor} (-1)^i \binom{n - i - 1}{i} (2C_k)^{n-2i-1}.
\]

The following result will be shown by using Binet’s formula.

**Theorem 3.3.** The following identity is valid for any natural number \( n \).

\[
B_n = \sum_{l=0}^{\left\lfloor \frac{n - 1}{2} \right\rfloor} \binom{n}{2l + 1} (\sqrt{8})^{2l} 3^{n-2l-1}.
\]
Proof. Recall that $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$. Setting $\lambda_1 = x + y$ and $\lambda_2 = x - y$ so that $x = 3$, $y = \sqrt{8}$ and therefore $\frac{y}{x} = \frac{\sqrt{8}}{3}$. Using binomial theorem, we have

$$\lambda_1^n - \lambda_2^n = (x + y)^n - (x - y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k (1 - (-1)^k).$$

Putting $k = 2l + 1$, we observe that for $k = 0$, the right side expression vanish, so $k = 1$ if $l = 0$ and $k = n$ if $l = \frac{n-1}{2}$. Therefore (8) reduces to

$$\frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} = B_n = \sum_{l=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2l+1} \left( \frac{\sqrt{8}}{3} \right)^{2l+1}.$$

It follows that

$$\frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} = B_n = \sum_{l=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2l+1} (\sqrt{8})^{2l} \frac{n-2l-1}{2^{2l+1}}.$$

This completes the proof.

We notice that, $B_0 = 0, B_1 = 1, B_2 = 6B_1 - B_0, \ldots, B_{k+2} = 6B_{k+1} - B_k$. Adding all these results, we get the following important identity.

$$B_{k+2} = 1 + 4 \sum_{i=0}^{k} B_i + 5B_{k+1}.$$  

4 Some congruence relations for balancing numbers

Ray has studied many congruence properties for balancing numbers and their related sequences [14, 15]. In [15], he applied some congruences identities to establish some divisibility properties of these numbers. In this section, we find some new results concerning congruences for balancing numbers.

Theorem 4.1. For any natural number $k$, $B_{2k} \equiv 0 \pmod{2}$ and $B_{2k} \equiv 0 \pmod{3}$.

Proof. Mathematical induction play the role to prove these results. Basis step is clear for the first part of the theorem as $B_0 = 0 \equiv 0 \pmod{2}$. Notice that $B_2 = 6 \equiv 0 \pmod{2}$. Assuming $B_{2m} \equiv 0 \pmod{2}$ for every $m \leq k$ and since $B_{k+1} = B_k B_{l+1} - B_{k-1} B_l$, we have

$$B_{2(m+1)} = B_{2m}B_3 - B_{2m-1}B_2 \equiv 0 \pmod{2}.$$  

Similarly the second part of the theorem can be proved.

The following corollary is an immediate consequence of Theorem 4.1

Corollary 4.2. For any natural number $k$, $B_{2k} \equiv 0 \pmod{6}$.
Theorem 4.3. For any natural number \( k \), \( B_{3k} \equiv 0 \pmod{5} \).

\[ (9) \]

Proof. The proof is analogous to Theorem 4.1.

Therefore by virtue of Theorem 4.1 and Theorem 4.2, we have the following result.

Corollary 4.4. For any natural number \( k \), \( B_{6k} \equiv 0 \pmod{10} \).

The following result can be easily shown by induction.

Theorem 4.5. For any natural number \( k \), \( B_{6k+2} \equiv 0 \pmod{2} \) and \( B_{6k} \equiv 0 \pmod{2} \).

Proof. By virtue of Theorem 3.3, we have

\[ B_n = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1} (\sqrt{8})^{2l} 3^{n-2l-1}. \]

Replacing \( n \) by \( 3k + 1 \), we get

\[ B_{3k+1} = \sum_{l=0}^{\lfloor \frac{3k}{2} \rfloor} \binom{3k + 1}{2l + 1} (\sqrt{8})^{2l} 3^{3k-2l}. \]

Notice that for every odd natural number \( k \), \( 5k + 4 \equiv 4 \pmod{5} \) and the result follows.

Theorem 4.7. If \( k \) is any odd natural number such that \( 4k + 1 \) is a prime, then \( B_{4k+1} \equiv 4k \pmod{4k+1} \).

Proof. Let \( k \in \mathbb{N} \) and \( k \) odd such that \( 4k + 1 \) is a prime. Putting \( n = 4k + 1 \) in (9) and since \( 8k + 1 \equiv 4k \pmod{4k+1} \), we have

\[ B_{4k+1} = \sum_{l=0}^{2k} \binom{4k + 1}{2l + 1} (\sqrt{8})^{2l} 3^{4k-2l} \equiv 8k + 1 \equiv 4k \pmod{4k+1}. \]

This ends the proof.

Similarly, putting \( n = 4k + 2 \) and \( n = 4k \) in (9) and since \( 8k + 2 \equiv 0 \pmod{4k+1} \) and \( 8k - 4 \equiv 4k - 5 \pmod{4k+1} \), we have the following results.

Theorem 4.8. If \( k \) is any odd natural number such that \( 4k + 1 \) is a prime, then \( B_{4k+2} \equiv 0 \pmod{4k+1} \).

Theorem 4.9. If \( k \) is any odd natural number such that \( 4k + 1 \) is a prime, then \( B_{4k} \equiv 4k - 5 \pmod{4k+1} \).

The following corollary is an immediate consequence of Theorem 4.8.

Corollary 4.10. If \( k \) is any odd natural number such that \( 4k + 1 \) is a prime, then \( B_{4k+2} \equiv 0 \pmod{4k+3} \).
References


