Balancing sequence contains no prime number

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Abstract: The study of prime number in any number sequence is crucial part. In recent year Panda and Behera introduced a new number sequence that is solutions of Diophantine equation $1 + 2 + 3 + \ldots + (n - 1) = (n + 1) + (n + 2) + \ldots + (n + r)$, where n and r are positive integers. The pairs (n, r) constitute a solution of above equation then n is called balancing number and r is the corresponding balancer. In this paper, we prove a main result that there is no prime number in the sequence of balancing numbers.
Keywords: Diophantine equation, Balancing numbers, Prime numbers, Cobalancing numbers.
AMS Classification: 11B83, 11A41.

1 Introduction

The concept of balancing number [1] is introduced by Panda and Behera in connection of Diophantine equation

$$1 + 2 + 3 \ldots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r) \tag{1}$$

for some $r \in Z^+$. Here r is called the balancer corresponding to balancing number n. For example, 6 is balancing number with balancer 2. Balancing numbers follow recurrence relation $B_{n+1} = 6B_n - B_{n-1}$. The study of balancing number can be seen in [2]. Properties of balancing numbers are very similar to Fibonacci numbers. Considering some balancing numbers like 6, 35 and 204, we observe that in sequence of balancing numbers, there is no prime number. In this study, we introduce a lemma, whose result is our main theorem, that there is no prime number in sequence of balancing numbers.

Panda and Ray modified the notion of balancing number to cobalancing number [3], in which natural N is called cobalancing number if

$$1+2+3\ldots+N=(N+1)+(N+2)+\cdots+(N+R)$$

for some natural number R, where R is called cobalancer of cobalancing number N. The first three cobalancing numbers are 2, 14 and 84 with cobalancers 1, 6 and 35, respectively. We also present a corollary on cobalancing numbers.

2 The proof

We first introduce the lemma to prove our main result.

Lemma 1.0. $8p^2+1$ is not a perfect square for all prime number p.

Proof. For $p=2$, $8\times 2^2+1=33$, which is not a perfect square.

Now, we use method of contradiction to prove our lemma. Let $8p^2+1$ be a perfect square for any prime number p, except 2. So,

$$8p^2 + 1 = a^2, \tag{2}$$

where $a \in N$, we also observe that a is an odd number.

Rearranging equation (2) we get,

$$2^3 p^2 = (a-1)(a+1).$$

By the virtue of fundamental theorem of arithmetic, the RHS of rearranged equation can be break into product of five prime numbers. We consider the different cases through which $(a-1)$ and $(a+1)$ can be break. Also we notice that, $(a-1)$ and $(a+1)$ are even terms.

- **Case 1:** Number of factors of $(a-1)$ is two and $(a+1)$ is three.
    - ○ **Sub case 1:** $(a-1)=2^2$ and $(a+1)=2p^2$. Solving this sub case we get $p^2=3$, which is an absurd.
    - ○ **Sub case 2:** $(a-1)=2p$ and $(a+1)=2^2p$. Solving this sub case we get $p=1$, which is an absurd.
    - ○ **Sub case 3:** $(a-1)=p^2$ and $(a+1)=2^3$. This sub case is not possible, since $(a-1)$ is even number but we have taken the assumption that $p\neq 2$ so, p^2 can never be even number, because the only even prime number is 2.
- **Case 2:** Number of factors of $(a-1)$ is three and $(a+1)$ is two.
    - ○ **Sub case 1:** $(a-1)=2p^2$ and $(a+1)=2^2$. Solving this sub case we get $p^2=1$, which is an absurd.
    - ○ **Sub case 2:** $(a-1)=2^2p$ and $(a+1)=2p$. Solving this sub case we get, $p=-1$, which is an absurd.
    - ○ **Sub case 3:** $(a-1)=2^3$ and $(a+1)=p^2$. This sub case is not possible, by the similar reason in case 1 and sub case 3.

- **Case 3:** Number of factors of $(a - 1)$ is one and $(a + 1)$ is four.
  - **Sub case 1:** $(a-1)=2$ and $(a+1)=2^2 p^2$. Solving this sub case gives $p^2=1$, which is an absurd.

  Other sub cases are not possible for case 3 because $(a - 1)$ is even number so, it cannot be equal to p except 2, but our assumption is that $p \neq 2$.
- **Case 4:** Number of factors of $(a - 1)$ is four and $(a + 1)$ is one.
  - **Sub case 1:** $(a-1)=2^2 p^2$ and $(a+1)=2$. Solving this sub case we get $p=0$, which is also an absurd.

  Also for this Case 4, other sub cases are not possible because $(a + 1)$ is even number so, it cannot be equal to p except 2, but our assumption is that $p \neq 2$.

So, all the possible cases give absurd results and contradict our assumption, hence proving the lemma. $\square$

Theorem 1.0. There is no prime number in the sequence of balancing numbers.
Proof. We use the method of contrapositive to prove our theorem. Solving (1) we get

$$n^2 = \frac{(n+r)(n+r+1)}{2},$$

and thus,

$$r = \frac{-(2n+1)+\sqrt{8n^2+1}}{2}.$$

The above expression states that when $8n^2 + 1$ is perfect square, then only n is a balancing number. So, considering the negation of this statement, i.e., if $8n^2 + 1$ is not a perfect square for any natural number n, then n is not a balancing number. By virtue of Lemma 1.0, $8p^2 + 1$ is not a perfect square for all prime numbers p, so for all prime numbers p is not a balancing number. This completes the proof. $\square$

Theorem 1.1. Every balancing number is cobalancer and every cobalancing number is balancer. [4]

Corollary 1.0. There is no cobalancer which is a prime number.
Proof. By Theorem 1.1 we get that every balancing number is cobalancer. Then by Theorem 1.0, we conclude that there is no prime number in balancing sequence. Hence, we proved the corollary. $\square$

Acknowledgement

References

[1] Behera, A., & Panda, G. K. (1999) On the square roots of triangular numbers, *Fibonacci Quarterly*, 37, 98–105.

[2] Ray, P. K. (2009) Balancing and Cobalancing Numbers, `http://www.nitrkl.ac.in/`, `http://ethesis.nitrkl.ac.in/2750/`.

[3] Panda, G. K., & Ray, P. K. (2005) Cobalancing numbers and cobalancers, *International Journal of Mathematics and Mathematical Sciences*, 8, 1189–1200.

[4] Panda, G. K., and Ray, P. K. (2011) Some links of balancing and cobalancing numbers with Pell and associated Pell numbers, *Bulletin of the Institute of Mathematics Academia Sinica (New Series)*, 6, 41–72.