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On certain logarithmic inequalities

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Abstract: We show how a logarithmic inequality from the book [1] is connected to means, and we offer new proofs, as well as refinements. We show that Karamata's [2] and Leach–Sholander's [3] inequality are in fact equivalent.
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1 Introduction

In the very interesting problem book by K. Hardy and K. S. Williams [1] (see 3., page 1) one can find the following logarithmic inequality:

$$\frac{\ln x}{x^3-1} < \frac{1}{3} \cdot \frac{x+1}{x^3+x}, \tag{1}$$

where $x > 0, x \neq 1$.

The proof of this surprisingly strong inequality is obtained in [1] by using a quite complicated study of auxiliary functions.

We wish to note in what follows, how inequality (1) is related to the famous logarithmic mean L, defined by

$$L(a,b) = \frac{a-b}{\ln a - \ln b}(a \neq b); L(a,a) = a, \tag{2}$$

where a and b are positive real numbers. We will show that, in terms of logarithmic mean, (1) is due in fact to J. Karamata [2]. For a survey of results on L and connected means, see e.g. [4, 5, 6].

Let $A(a,b) = \dfrac{a+b}{2}$, $G(a,b) = \sqrt{ab}$ denote the classical arithmetic, resp. geometric mean of a and b. It is well known that, the logarithmic mean separates the geometric and arithmetic mean:

$$G < L < A, \tag{3}$$

where $G = G(a,b)$, etc. and $a \neq b$. For the history of this inequality and new proofs, see [5, 15, 17, 18, 19].

As inequality (3) is important in many fields of mathematics, (see e.g. [9, 12, 15]), the following famous refinement of left side of (3), due to Leach and Sholander [3] should be mentioned

$$\sqrt[3]{G^2 \cdot A} < L. \tag{4}$$

Now, let us introduce the following mean K by

$$K(a,b) = \frac{a\sqrt[3]{b} + b\sqrt[3]{a}}{\sqrt[3]{b} + \sqrt[3]{a}}. \tag{5}$$

Letting $x = \sqrt[3]{\dfrac{a}{b}}$ $(a \neq b)$, inequality (1) can be written, by using (2) and (5):

$$L(a,b) > K(a,b). \tag{6}$$

This inequality is due to Karamata [2].

We will show that inequality (6) refines (4). Also, we will give new proof and refinements to this inequality.

2 Main results

The first result shows that (6) is indeed a refinement of (4):

Theorem 1. *One has*

$$L > K > \sqrt[3]{G^2 A}. \tag{7}$$

Proof. We have to prove the second inequality of (7), i.e.,

$$\frac{a\sqrt[3]{b} + b\sqrt[3]{a}}{\sqrt[3]{b} + \sqrt[3]{a}} > \sqrt[3]{ab \cdot \left(\frac{a+b}{2}\right)}. \tag{8}$$

Putting $a = u^3, b = v^3$, this inequality becomes

$$\frac{u^3 v + v^3 u}{u+v} > uv \cdot \sqrt[3]{\frac{u^3+v^3}{2}},$$

or after elementary transformations:

$$2(u^2+v^2)^3 > (u+v)^3 \cdot (u^3+v^3). \tag{9}$$

This inequality, which is interesting in itself, can be proved by algebraic computations; here we present an analytic approach, used also in our paper [11]. By logarithmation, the inequality becomes

$$\ln 2 + 3\ln(u^2+v^2) - 3\ln(u+v) - \ln(u^3+v^3) = f(u) > 0. \tag{10}$$

Suppose $u > v$. Also, for simplicity one could take $v = 1$ (since (9) is homogeneous). Then one has $f'(u) = \dfrac{6u}{u^2+1} - \dfrac{3}{u+1} - \dfrac{3u^2}{u^3+1} = \dfrac{(u-1)^3}{(u^2+1)(u^3+1)} > 0$, after elementary computations, which we omit here. Thus $f(u) > f(1) = 0$, and the result follows. □

Remark 1. *Inequality $K > \sqrt[3]{G^2A}$ has been discovered by the author in 2003 [10]. For the extensions of* (9)*, see [10] and [16].*

Theorem 2. *Inequality $L > K$ is equivalent to inequality*

$$L > \frac{3AG}{2A+G} \tag{11}$$

Proof. By letting $a = u^3, b = v^3$ the inequality $L(a,b) > K(a,b)$ becomes the equivalent inequality $L(u^3,v^3) > K(u^3,v^3)$. Now, remark that $L(u^3,v^3) = L(u,v)\cdot\frac{u^2+uv+v^2}{3}$ and $K(u^3,v^3) = \frac{uv(u^2+v^2)}{u+v}$, so we get the relation

$$L(u,v) > \frac{3uv(u^2+v^2)}{(u+v)(u^2+uv+v^2)} \tag{12}$$

Let now $u = \sqrt{p}, v = \sqrt{q}$ in (12), with $p \neq q$ positive real numbers. Remarking that $L(\sqrt{p},\sqrt{q}) = \frac{2}{\sqrt{u}+\sqrt{v}} \cdot L(p,q)$, after certain computations, (12) becomes

$$L(p,q) > \frac{3\sqrt{pq}(p+q)}{p+q+\sqrt{pq}} \tag{13}$$

As $\sqrt{pq} = G(p,q), p+q = 2A(p,q)$; inequality (13) may be written as

$$L > \frac{3AG}{2A+G}, \tag{14}$$

where $L = L(p,q)$ etc. Clearly, this inequality is independents of the variables p and q, and could take $L = L(a,b), A = A(a,b), G = G(a,b)$ in inequality (14). This proves Theorem 2. □

Remark 2. *For inequalities related to* (11)*, see also [11].*

Now, the surprise is that, though (6) is stronger than (4), inequality (4) implies inequality (6)!

Theorem 3. *One has*

$$L > \sqrt[3]{G^2A} > \frac{3AG}{2A+G}. \tag{15}$$

Proof. The first inequality of (15) is the Leach–Sholander inequality (4).

Now, remark that $\sqrt[3]{G^2A} =$ geometric mean of: G, G and $A = \sqrt[3]{G \cdot G \cdot A}$, which is greater than the harmonic mean of these three numbers:

$$\frac{3}{\frac{1}{G}+\frac{1}{G}+\frac{1}{A}} = \frac{3}{\frac{2}{G}+\frac{1}{A}} = \frac{3AG}{2A+G}.$$

Therefore, inequality (15) follows. □

Theorem 4. *One has*

$$L > \sqrt[3]{\left(\frac{A+G}{2}\right)^2 \cdot G} > \frac{3G(A+G)}{A+5G} > \frac{3AG}{2A+G}. \tag{16}$$

Proof. The first inequality of (16) is a refinement of (4), and is due to the author [7]. See also [13].

The second inequality of (16) follows by the same argument as the proof of Theorem 3: the geometric mean of the numbers $\frac{A+G}{2}, \frac{A+G}{2}, G$ is greater than their harmonic mean, which is

$$\frac{3}{\frac{2}{A+G}+\frac{2}{A+G}+\frac{1}{G}} = \frac{3G(A+G)}{5G+A}.$$

Finally, the last inequality is equivalent, after some computations with $A^2 - 2AG + G^2 > 0$, or $(A-G)^2 > 0$. □

Remark 3. *Connections of L with other means are studied in papers [6, 8, 14].*

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