On certain logarithmic inequalities

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Abstract: We show how a logarithmic inequality from the book [1] is connected to means, and we offer new proofs, as well as refinements. We show that Karamata’s [2] and Leach–Sholander’s [3] inequality are in fact equivalent.

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1 Introduction

In the very interesting problem book by K. Hardy and K. S. Williams [1] (see 3., page 1) one can find the following logarithmic inequality:

\[
\frac{\ln x}{x^3 - 1} < \frac{1}{3} \cdot \frac{x + 1}{x^3 + x},
\]

where \(x > 0, x \neq 1\).

The proof of this surprisingly strong inequality is obtained in [1] by using a quite complicated study of auxiliary functions.

We wish to note in what follows, how inequality (1) is related to the famous logarithmic mean \(L\), defined by

\[
L(a, b) = \frac{a - b}{\ln a - \ln b} (a \neq b); L(a, a) = a,
\]

where \(a\) and \(b\) are positive real numbers. We will show that, in terms of logarithmic mean, (1) is due in fact to J. Karamata [2]. For a survey of results on \(L\) and connected means, see e.g. [4, 5, 6].
Let \( A(a, b) = \frac{a + b}{2} \), \( G(a, b) = \sqrt{ab} \) denote the classical arithmetic, resp. geometric mean of \( a \) and \( b \). It is well known that, the logarithmic mean separates the geometric and arithmetic mean:

\[
G < L < A, \tag{3}
\]

where \( G = G(a, b) \), etc. and \( a \neq b \). For the history of this inequality and new proofs, see [5, 15, 17, 18, 19].

As inequality (3) is important in many fields of mathematics, (see e.g. [9, 12, 15]), the following famous refinement of left side of (3), due to Leach and Sholander [3] should be mentioned

\[
3\sqrt{G^2} \cdot A < L. \tag{4}
\]

Now, let us introduce the following mean \( K \) by

\[
K(a, b) = \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b} + \sqrt{a}}. \tag{5}
\]

Letting \( x = 3\sqrt[3]{\frac{a}{b}} \) \( (a \neq b) \), inequality (1) can be written, by using (2) and (5):

\[
L(a, b) > K(a, b). \tag{6}
\]

This inequality is due to Karamata [2].

We will show that inequality (6) refines (4). Also, we will give new proof and refinements to this inequality.

## 2 Main results

The first result shows that (6) is indeed a refinement of (4):

**Theorem 1.** One has

\[
L > K > \sqrt[3]{G^2} \cdot A. \tag{7}
\]

**Proof.** We have to prove the second inequality of (7), i.e.,

\[
\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b} + \sqrt{a}} > \sqrt[3]{ab \cdot \left( \frac{a + b}{2} \right)}. \tag{8}
\]

Putting \( a = u^3, b = v^3 \), this inequality becomes

\[
\frac{u^3v + v^3u}{u + v} > uv \cdot \sqrt[3]{\frac{u^3 + v^3}{2}},
\]

or after elementary transformations:

\[
2(u^2 + v^2)^3 > (u + v)^3 \cdot (u^3 + v^3). \tag{9}
\]
This inequality, which is interesting in itself, can be proved by algebraic computations; here we present an analytic approach, used also in our paper [11]. By logarithmation, the inequality becomes

\[ \ln 2 + 3 \ln(u^2 + v^2) - 3 \ln(u + v) - \ln(u^3 + v^3) = f(u) > 0. \] (10)

Suppose \( u > v \). Also, for simplicity one could take \( v = 1 \) (since (9) is homogeneous). Then one has \( f'(u) = \frac{6u}{u^2 + 1} - 3 \frac{u^2 - 1}{u^2 + 1} = \frac{(u - 1)^3}{(u^2 + 1)(u^3 + 1)} > 0 \), after elementary computations, which we omit here. Thus \( f(u) > f(1) = 0 \), and the result follows.

Remark 1. Inequality \( K > \sqrt[3]{G^2 A} \) has been discovered by the author in 2003 [10]. For the extensions of (9), see [10] and [16].

Theorem 2. Inequality \( L > K \) is equivalent to inequality

\[ L > \frac{3AG}{2A + G}. \] (11)

Proof. By letting \( a = u^3, b = v^3 \) the inequality \( L(a, b) > K(a, b) \) becomes the equivalent inequality \( L(u^3, v^3) > K(u^3, v^3) \). Now, remark that \( L(u^3, v^3) = L(u, v) \cdot \frac{u^2 + uv + v^2}{uv(u^2 + v^2)} \) and \( K(u^3, v^3) = \frac{uv(u^2 + v^2)}{u + v} \), so we get the relation

\[ L(u, v) > \frac{3uv(u^2 + v^2)}{(u + v)(u^2 + uv + v^2)}. \] (12)

Let now \( u = \sqrt{p}, v = \sqrt{q} \) in (12), with \( p \neq q \) positive real numbers. Remarking that \( L(\sqrt{p}, \sqrt{q}) = \frac{2}{\sqrt{p} + \sqrt{q}} \cdot L(p, q) \), after certain computations, (12) becomes

\[ L(p, q) > \frac{3\sqrt{pq}(p + q)}{p + q + \sqrt{pq}}. \] (13)

As \( \sqrt{pq} = G(p, q), p + q = 2A(p, q) \); inequality (13) may be written as

\[ L > \frac{3AG}{2A + G}, \] (14)

where \( L = L(p, q) \) etc. Clearly, this inequality is independents of the variables \( p \) and \( q \), and could take \( L = L(a, b), A = A(a, b), G = G(a, b) \) in inequality (14). This proves Theorem 2. 

Remark 2. For inequalities related to (11), see also [11].

Now, the surprise is that, though (6) is stronger than (4), inequality (4) implies inequality (6)!
Theorem 3. One has

\[ L > \sqrt[3]{G^2A} > \frac{3AG}{2A+G}. \]  \tag{15}

Proof. The first inequality of (15) is the Leach–Sholander inequality (4).

Now, remark that \( \sqrt[3]{G^2A} = \text{geometric mean of: } G, G \) and \( A = \sqrt{G \cdot G \cdot A} \), which is greater than the harmonic mean of these three numbers:

\[ \frac{3}{\frac{1}{G} + \frac{1}{G} + \frac{1}{A}} = \frac{3}{\frac{2}{G} + \frac{1}{A}} = \frac{3AG}{2A+G}. \]

Therefore, inequality (15) follows. \qed

Theorem 4. One has

\[ L > \sqrt[3]{\left(\frac{A+G}{2}\right)^2} \cdot G > \frac{3G(A+G)}{A+5G} > \frac{3AG}{2A+G}. \]  \tag{16}

Proof. The first inequality of (16) is a refinement of (4), and is due to the author [7]. See also [13].

The second inequality of (16) follows by the same argument as the proof of Theorem 3: the geometric mean of the numbers \( \frac{A+G}{2}, \frac{A+G}{2}, G \), is greater than their harmonic mean, which is

\[ \frac{3}{\frac{2}{A+G} + \frac{2}{A+G} + \frac{1}{G}} = \frac{3G(A+G)}{5G+A}. \]

Finally, the last inequality is equivalent, after some computations with \( A^2 - 2AG + G^2 > 0 \), or \( (A - G)^2 > 0 \). \qed

Remark 3. Connections of \( L \) with other means are studied in papers [6, 8, 14].

References


