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On the irrationality of $\sqrt{N}$

József Sándor and Edith Egri

Babeş–Bolyai University
Cluj–Napoca, Romania
e-mails: jsandor@math.ubbcluj.ro,
egriedith@gmail.com

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Abstract: We offer a new proof of the classical fact that $\sqrt{N}$ is irrational, when N is not a perfect square.
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1 Introduction

Since the times of Euclid, there appeared many proofs of the irrationality of $\sqrt{2}$ or generally $\sqrt{N}$, where N is not a perfect square.

In what follows we shall offer another proof, which seems to be new. The main idea is contained in Marty Ross' paper [1].

2 The proof

Marty Ross [1] discovered the following very interesting new proof of irrationality of $\sqrt{2}$: Suppose that $\sqrt{2} = \dfrac{m}{n}$, with m and n positive integers. Then clearly one has $n < m < 2n$. Remark that

$$\sqrt{2} = \frac{\sqrt{2} \cdot (\sqrt{2} - 1)}{\sqrt{2} - 1} = \frac{2 - \sqrt{2}}{\sqrt{2} - 1} = \frac{2 - \frac{m}{n}}{\frac{m}{n} - 1} = \frac{2n - m}{m - n} = \frac{m_1}{n_1}.$$

Here $n_1 > 0$ and $0 < m_1 < m$ by the inequalities $n < m < 2n$. This method can be continued infinitely often, and we will obtain a strictly decreasing sequence (m_k) of positive integers. In other words, we have Fermat's method of "infinite descend". Such a sequence cannot exist, so $\sqrt{2}$ is irrational.

Now we will generalize the above idea to prove the irrationality of $\sqrt{N}$, when N is not a perfect square. Since N is not a perfect square, there exist two consecutive perfect squares separating $N : k^2 < N < (k+1)^2$; i.e. $k < \sqrt{N} < k+1$. Suppose that $\sqrt{N}$ is rational, with smallest positive denominator m: $\sqrt{N} = \dfrac{m}{n}$. By the identity $\sqrt{N} = \sqrt{N} \cdot \dfrac{\sqrt{N} - k}{\sqrt{N} - k} = \dfrac{N - k\sqrt{N}}{\sqrt{N} - k}$ we get $\sqrt{N} = \dfrac{N - k\frac{m}{n}}{\frac{m}{n} - k} = \dfrac{Nn - km}{m - kn} = \dfrac{m_1}{n_1}$. Here $n_1 > 0$ as $k < \dfrac{m}{n}$. On the other hand, $Nn - km = m_1 < m$, since this is equivalent to $N < (k+1)\dfrac{m}{n}$; i.e. $\dfrac{m^2}{n^2} < (k+1)\dfrac{m}{n}$, so $\dfrac{m}{n} < k+1$, which is true by assumption. Therefore, $0 < m_1 < m$; contradicting the minimality of m. This finishes the proof of irrationality of $\sqrt{N}$.

Remark 1. *As we can remark, this proof is not based on the unique factorization theorem (i.e. basic theorem of arithmetics), but on the fact that the set* $\mathbb{N}$ *of positive integers is well-ordered, i.e. every bounded below subset of* $\mathbb{N}$ *has a smallest element.*

This proof is so straightforward and natural, that we think, it should have a place in the "Book" imagined by Erdős [2]. Also, the method presented here has a motivation for teaching purposes.

References

[1] Ross, M. (2004) Irrational thoughts, *Math. Gazette*, March, 2004, pp. 68–78.

[2] Aigner, M., & Ziegler, M. G. (2014) *Proofs from the Book*, Fifth ed., Springer-Verlag Berlin Heidelberg.

Note added in proof

After completing this note, we incidentally discovered that the proof by M. Ross can be found essentially in the famous book: Whittaker, E. T., & Watson, G. N. (1920) *A Course of Modern Analysis*, 3rd edition, Cambridge Univ. Press.