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Some characteristics of the Golden Ratio family

J. V. Leyendekkers[1] and A. G. Shannon[2, 3]

[1] Faculty of Science, The University of Sydney, NSW 2006, Australia
[2] Faculty of Engineering & IT, University of Technology, Sydney, NSW 2007, Australia
[3] Campion College, PO Box 3052, Toongabbie East, NSW 2146, Australia
e-mails: t.shannon@campion.edu.au, Anthony.Shannon@uts.edu.au

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Abstract: Research into properties of generalizations of the Golden Ratio has considered various forms of extreme and mean ratios. This is considered here too within the framework of a family of surds, $\frac{1}{2}\sqrt{1+a}$, and generalized Fibonacci numbers, $F_n(a)$, with the ordinary Fibonacci numbers being the particular case when $a = 5$.

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1 Introduction

The classical Golden Ratio, known since the time of Euclid [8] is the first member of a family of surds, $\frac{1}{2}\sqrt{1+a}$, the members of which have many similar properties [5, 7], which we propose to explore further in this paper.

Other such research has included that of Kapur [4] who used the fact that the Golden Ratio is related to the division of a line into two parts to then consider the division of a line into n parts as a way of generalising the Golden Ratio. Analogously, other generalizations have considered a 'silver mean' $\frac{2a+b}{a} = \frac{a}{b}$ related to the Pell numbers in a manner similar to the golden mean $\frac{a+b}{a} = \frac{a}{b}$, related to the Fibonacci numbers, and generalizations of the Fibonacci continued fraction [9]. More recently, Crăciun *et al* [3] have introduced a generalized Golden Ratio as a fixed point of an operator defined by an arbitrary mean, which satisfies certain conditions.

In general, each member of the family satisfies

$$\varphi_a = \frac{1+\sqrt{a}}{2}, \tag{1.1}$$

in which $a = 4r_1 + 1, a \in \bar{1}_4 \subset z_4$ (Table 1). The associated Fibonacci sequences satisfy [1]:

$$F_{n+1} = F_n + r_1 F_{n-1}. \quad (1.2)$$

Row $r_i \downarrow$	Class $i \rightarrow$	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$	Comments
0		0	1	2	3	• $N = 4r_i + i$
1		4	5	6	7	• even $\bar{0}_4, \bar{2}_4$
2		8	9	10	11	• $\left(N^n, N^{2n}\right) \in \bar{0}_4$
3		12	13	14	15	• odd $\bar{1}_4, \bar{3}_4$ **;** $N^{2n} \in \bar{1}_4$

Table 1. Classes and rows for Z_4

2 Extreme and mean ratio

The Golden Ratio appears in the Elements of Euclid in several places [8]. Thus, if we take a line *AB* and divide by a point *C*, Euclid's definition of extreme and mean ratio is "(larger segment, *AC*) over (shorter segment, *CB*) is equal to (whole line, *AB*) over (larger segment, *AC*); that is,

$$AC/CB = AB/AC, \quad (2.1)$$

$$AC^2 = CB \times AB, \quad (2.2)$$

$$AB = AC + CB. \quad (2.3)$$

If $AC = \varphi_a$, then since

$$AC^2 = \varphi_a^2 = \varphi_a + r_1, \quad (2.4)$$

from Equations (2.1) to (2.4) we have:

$$\begin{aligned} AB &= AC + AC^2/AB \\ AB^2 &= AB \times AC + AC^2 \\ &= AB \times \varphi_a + AC^2 \\ 0 &= AB^2 - \varphi_a AB - AC^2 \\ &= AB^2 - \varphi_a AB - \varphi_a^2 \end{aligned} \quad (2.5)$$

which has a solution

$$\begin{aligned} AB &= \tfrac{1}{2}\left(\varphi_a + \sqrt{\varphi_a^2 + 4\varphi_a^2}\right) \\ &= \tfrac{1}{2}\left(\varphi_a + \sqrt{5\varphi_a^2}\right) \\ &= \varphi_a\left(\frac{1+\sqrt{5}}{2}\right) \\ AB &= \varphi_a \varphi_5 \end{aligned} \quad (2.6)$$

and

$$CB = \varphi_a(\varphi_5 - 1). \tag{2.7}$$

For example, when $a = 5$,

$$AB = \varphi_5^2 = \varphi_5 + 1. \tag{2.8}$$

We observe in Table 2 that the change in AB becomes increasingly smaller and AB reaches a constant value as $\varphi_{a+4} - \varphi_a$ approaches zero.

a	φ_a	AB
5	1.6180339	2.6180339
9	2.0000000	3.2360618
13	2.3027756	3.7259689
17	2.5615528	4.1446876
21	2.7912878	4.5163982
25	3.0000000	4.8541017
29	3.1925824	5.1657065
33	3.3722813	5.4564654
37	3.5413812	5.7300748

Table 2. AB and φ_a

3 Simson's identity

When $a = 5$, we have the well-known Simson's identity:

$$F_n^2 = F_{n+1}F_{n-1} + (-1)^{n-1}. \tag{3.1}$$

The question naturally arises does this identity extend to the whole Golden Ratio family [5]? The answer is 'yes' in the form (Tables 3 and 4):

$$F_n^2 = F_{n+1}F_{n-1} + (-r_1)^{n-1}. \tag{3.2}$$

n	F_n^2	F_{n+1}	F_{n-1}	$F_{n+1}F_{n-1}$	$(-r_1)^{n-1}$
2	1	4	1	4	–3
3	16	7	1	7	9
4	49	19	4	76	–27
5	361	40	7	280	81
6	1600	97	19	1843	–243
7	9409	217	40	8880	729

Table 3. $a = 13$, $r_1 = 3$; $F_n^2 = F_{n+1}F_{n-1} + (-3)^{n-1}$.

n	F_n^2	F_{n+1}	F_{n-1}	$F_{n+1}F_{n-1}$	$(-r_1)^{n-1}$
2	1	5	1	5	–4
3	25	9	1	9	16
4	81	29	5	145	–64
5	841	65	9	585	256
6	4225	181	29	5249	–1024
7	32761	441	65	28665	4096

Table 4. $a = 17$, $r_1 = 4$; $F_n^2 = F_{n+1}F_{n-1} + (-4)^{n-1}$.

4 Reciprocals

The reciprocals of $(1 + \varphi_a)$ can be found from

$$R = \frac{r_1 - 2}{1 + \varphi_a} \tag{4.1}$$

in which

$$R = \varphi_a - 2. \tag{4.2}$$

Some examples are given in Table 4. Table 5 illustrates the consistency among the Golden Ratio family members.

a	r_1	$(1+\varphi_a)$	$(1+\varphi_a)^{-1}$	$\frac{r_1-2}{1+\varphi_a}$
5	1	2.6180339	0.3819660	–0.3819660
13	3	3.3027756	0.3027756	0.3027756
17	4	3.5615528	0.2807764	0.5615528
21	6	3.7912878	0.2637626	0.7912878
29	7	4.1925824	0.2385164	1.1925820
33	8	4.3722813	0.2287134	1.3722804
37	9	4.5413812	0.2201973	1.5413811
41	10	4.7015621	0.2126952	1.7015621
45	11	4.8541010	0.2060113	1.8541017
53	13	5.1400544	0.1945504	2.1400544

Table 5. Reciprocals of $(1+\varphi_a)$

5. Golden Ratio circles

Since $a \in \bar{1}_4$, a can be a sum of squares (Table 6). Thus

$$a = x^2 + y^2, \tag{5.1}$$

where x, y may be calculated from [6]:

$$x, y = \frac{A \pm \sqrt{2a - A^2}}{2} \tag{5.2}$$

with x odd and y even, and

$$A = x + y. \tag{5.1}$$

If a has factor in $\bar{3}_4$, then there could be no integers x, y [6]. From Equations (5.1 and 5.2) $\sqrt{a}$ may be the radius of a circle in which x and y are integers as in Figure 1 and Table 6.

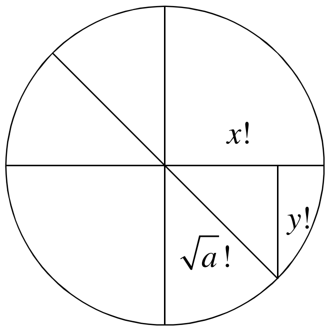

Figure 1. Equation (5.1)

a	$r = \sqrt{a} = 2\varphi_a - 1$	x	y
5	2.2360670	1	2
13	3.0605551	3	2
17	4.1231056	1	4
25	5.0000000	3	4
29	5.3851648	5	2
37	6.0827625	1	6

Table 6.

An infinity of circles with radius ($2\varphi_a - 1$), associated with the Golden Ratio, may thus be formed. This links φ_a to a variety of geometric figures such as cylinders, right circular cones, spheres and so on [2, 10].

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