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Right circulant matrices with ratio of the elements of Fibonacci and geometric sequence

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Abstract: We introduce the right circulant matrices with ratio of the elements of Fibonacci and geometric sequence. Furthermore, we investigate their eigenvalues, determinant, Euclidean norm, and inverse.
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1 Introduction

It has been a tradition to use number sequences as entries for right circulant matrices. Works on these special matrices normally would concentrate on the investigation of their eigenvalues, determinants, Euclidean norms, spectral norms and inverses. Recent works on this study include that of Bahsi and Solak [**?**] and Bueno [**?**], [**?**].

Bahsi and Solak used arithmetic sequence as entries. They provided the explicit formulas for the determinant, eigenvalues, Euclidean and spectral norms, and inverses of these special type of right circulant matrices. They also investigated the Euclidean and spectral norms of the inverses of these matrices. Bueno also did the same in [**?**] (using geometric sequence) and in [**?**] (using Fibonacci sequence).

In this study, we will use the sequence $\{s_k\}_{k=0}^{+\infty}$ where $s_k = \frac{F_k}{ar^k}$ is the ratio of the kth Fibonacci number and the kth element of geometric sequence. Hence, the right circulant matrix takes the form

$$C_R(\vec{s}) = \begin{pmatrix} s_0 & s_1 & s_2 & \dots & s_{n-2} & s_{n-1} \\ s_{n-1} & s_0 & s_1 & \dots & s_{n-3} & s_{n-2} \\ s_{n-2} & s_{n-1} & s_0 & \dots & s_{n-4} & s_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_2 & s_3 & s_4 & \dots & s_0 & s_1 \\ s_1 & s_2 & s_3 & \dots & s_{n-1} & s_0 \end{pmatrix}.$$

As a continuing tradition on the study of right circulant matrices, we will derive the explicit forms of the eigenvalues, determinant, Euclidean norm and inverse of the matrix $C_R(\vec{s})$.

For the rest of the paper we will use $|C_R(\vec{s})|$, $||C_R(\vec{s})||_E$ and $C_R^{-1}(\vec{s})$ to denote the determinant, the Euclidean norm and the inverse of $C_R(\vec{s})$.

2 Main results

We now present the main results of our study.

Theorem 2.1. *The eigenvalues of $C_R(\vec{s})$ are given by*

$$\lambda_m = \frac{-rF_n - (F_{n-1} - r^n)\omega^{-m}}{ar^{n-1}(r - \phi\omega^{-m})(r - \psi\omega^{-m})} \tag{1}$$

where $m = 0,\ 1,\ \dots,\ n-1$, $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$ and $\omega = e^{2\pi i/n}$.

Proof: The discrete Fourier transform is used to solve the eigenvalues of right circulant matrices. Solving for the eigenvalues of $C_R(\vec{s})$ yields

$$\begin{aligned} \lambda_m &= \sum_{k=0}^{n-1} \frac{F_k}{ar^k}\omega^{-mk} \\ &= \sum_{k=0}^{n-1} \frac{\phi^k - \psi^k}{ar^k}\omega^{-mk} \end{aligned}$$

where $m = 0,\ 1,\ \dots,\ n-1$, $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$ and $\omega = e^{2\pi i/n}$.

Continuing we have

$$\begin{aligned} \lambda_m &= \frac{1}{a\sqrt{5}}\left[\sum_{k=0}^{n-1}\left(\frac{\phi\omega^{-m}}{r}\right)^k - \left(\frac{\psi\omega^{-m}}{r}\right)^k\right] \\ &= \frac{1}{a\sqrt{5}}\left[\frac{1-(\phi/r)^n}{1-\phi\omega^{-m}/r} - \frac{1-(\psi/r)^n}{1-\psi\omega^{-m}/r}\right] \\ &= \frac{1}{ar^{n-1}\sqrt{5}}\left[\frac{r^n-\phi^n}{r-\phi\omega^{-m}} - \frac{r^n-\psi^n}{r-\psi\omega^{-m}}\right] \\ &= \frac{-r(\phi^n-\psi^n) + r^n\omega^{-m}(\phi-\psi) - \omega^{-m}(\phi^{n-1}-\psi^{n-1})}{ar^{n-1}\sqrt{5}(r-\phi\omega^{-m})(r-\psi\omega^{-m})} \end{aligned}$$

$$
\begin{aligned}
&= \frac{-rF_n + r^n F_1 \omega^{-m} - F_{n-1}\omega^{-m}}{ar^{n-1}\sqrt{5}(r-\phi\omega^{-m})(r-\psi\omega^{-m})} \\
&= \frac{-rF_n - (F_{n-1} - r^n)\omega^{-m}}{ar^{n-1}(r-\phi\omega^{-m})(r-\psi\omega^{-m})}.
\end{aligned}
$$

□

Theorem 2.2.

$$
|C_R(\vec{s})| = \frac{(-1)^n r^n F_n - (r^n - F_{n-1})^n}{a^n r^{n(n-1)}(r^{2n} - L_n + (-1)^n)} \tag{2}
$$

where $L_n = \phi^n + \psi^n$*, the* n*th Lucas number.*

Proof: Recall that the determinant is the product of the eigenvaluesm so we have

$$
|C_R(\vec{s}))| = \prod_{m=0}^{n-1} \frac{-rF_n - (F_{n-1} - r^n)\omega^{-m}}{ar^{n-1}(r-\phi\omega^{-m})(r-\psi\omega^{-m})}
$$

Note that for any x and y,

$$
\prod_{m=0}^{n-1}(x - y\omega^{-m}) = x^n - y^n
$$

Using this relationship results to

$$
\begin{aligned}
|C_R(\vec{s})| &= \frac{(-1)^n r^n F_n^n - (r^n - F_{n-1})^n}{a^n r^{n(n-1)}(r^{2n} - r^n(\phi^n + \psi^n) + (-1)^n)} \\
&= \frac{(-1)^n r^n F_n - (r^n - F_{n-1})^n}{a^n r^{n(n-1)}(r^{2n} - L_n + (-1)^n)}
\end{aligned}
$$

□

Theorem 2.3.

$$
||C_R(\vec{s})||_E = \frac{\sqrt{\frac{n}{5}[A_n - 2B_n]}}{|ar^{n-1}|} \tag{3}
$$

where $A_n = \frac{2r^{2n+2} - r^2 L_{2n} + r^2 L_2 + L_{2n-2}}{r^4 - r^2 L_2 + 1}$ *and* $B_n = \frac{r^{2n} - (-1)^n}{r^2+1}$.

Proof:

$$
\begin{aligned}
||C_R(\vec{s})||_E &= \sqrt{n \sum_{k=0}^{n-1} \left| \frac{\phi^k - \psi^k}{ar^k\sqrt{5}} \right|^2} \\
&= \sqrt{\frac{n}{5a^2} \sum_{m=0}^{n-1} \left[\frac{\phi^{2k} + \psi^{2k} - 2(-1)^k}{r^{2k}} \right]} \\
&= \sqrt{\frac{n}{5a^2} \left[\frac{1 - \phi^{2n}/r^{2n}}{1 - \phi^2/r^2} + \frac{1 - \psi^{2n}/r^{2n}}{1 - \psi^2/r^2} - \frac{2(1 - (-r^{-2}))^n}{1 + r^{-2}} \right]} \\
&= \sqrt{\frac{n}{5a^2 r^{2(n-1)}} \left[\frac{r^{2n} - \phi^{2n}}{r^2 - \phi^2} - \frac{r^{2n} - \psi^{2n}}{r^2 - \psi^2} - 2\left(\frac{r^{2n} - (-1)^n}{r^2+1} \right) \right]}
\end{aligned}
$$

Simplifying further, we obtain

$$||C_R(\vec{s})||_E = \frac{\sqrt{\frac{n}{5}\left[\frac{2r^{2n+2}-r^2L_{2n}+r^2L_2+L_{2n-2}}{r^4-r^2L_2+1} - 2\left(\frac{r^{2n}-(-1)^n}{r^2+1}\right)\right]}}{|ar^{n-1}|}$$

□

Theorem 2.4.

$$C_R^{-1} = C_R\left(t_0,\ t_1,\ \ldots,\ t_{n-1}\right) \tag{4}$$

where

$$t_k = -\frac{ar^{n-1}}{n}\sum_{m=0}^{n-1}\frac{(r-\phi\omega^{-m})(r-\psi\omega^{-n})}{rF_n+(F_{n-1}-r^n)\omega^{-m}} \tag{5}$$

where $k = 0,\ 1,\ \ldots,\ n-1$ *and provided that* $(r^n - F_{n-1})\omega^{-m} \neq rF_n$.

Proof: The first row entries of the inverse of a right circulant matrix is just the inverse discrete Fourier transform of the reciprocal of the eigenvalues. Hence, we will have

$$t_k = \frac{1}{n}\sum_{m=0}^{n-1}\lambda_m^{-1}\omega^{mk}$$

where $k = 0,\ 1,\ \ldots,\ n-1$.

Note that for the inverse to exist, all eigenvalues should be non-zero. So for all m, $(r^n - F_{n-1})\omega^{-m} \neq rF_n$.

The above equation results to

$$t_k = -\frac{ar^{n-1}}{n}\sum_{m=0}^{n-1}\frac{(r-\phi\omega^{-m})(r-\psi\omega^{-n})}{rF_n+(F_{n-1}-r^n)\omega^{-m}}$$

□

3 Conclusion

In summary, we have obtained the closed form of the eigenvalues, determinant, Euclidean norm and the inverse of $C_R(\vec{s})$. Furthermore, we have the following observations:

- The eigenvalues, the determinant and the inverse of $C_R(\vec{s})$ are all functions of F_n, F_{n-1}, ϕ, ψ, a, r and n.
- The Euclidean norm of $C_R(\vec{s})$ is a function of L_2, L_{2n}, L_{2n-2}, a, r and n.

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