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Embedding the unitary divisor meet semilattice in a lattice

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Abstract: A positive divisor d of a positive integer n is said to be a unitary divisor of n if $(d, n/d) = 1$. The set of positive integers is a meet semilattice under the unitary divisibility relation but not a lattice since the least common unitary multiple (lcum) does not always exist. This meet semilattice can be embedded to a lattice; two such constructions have hitherto been presented in the literature. Neither of them is distributive nor locally finite. In this paper we embed this meet semilattice to a locally finite distributive lattice. As applications we consider semimultiplicative type functions, meet and join type matrices and the Möbius function of this lattice.
Keywords: Unitary divisor, Meet semilattice, Distributive lattice, Semimultiplicative function, Meet matrix, Möbius function.
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1 Introduction

It is well known that the set $\mathbb{Z}_+$ of positive integers is a poset under the usual divisibility relation. It is likewise well known that the gcd and the lcm operations serve as the meet and the join on this poset. Thus $\mathbb{Z}_+$ is a lattice under the usual divisibility relation, known as the divisor lattice. This lattice is distributive.

A divisor $d \in \mathbb{Z}_+$ of $n \in \mathbb{Z}_+$ is said to be a unitary divisor of n and is denoted by $d \| n$ if $(d, n/d) = 1$. For example, the unitary divisors of 200 $(= 2^3 5^2)$ are $1, 8, 25, 200$. The unitary divisors of a prime power p^a are 1 and p^a. A general formula for the unitary divisors of $n = \prod_{p\in\mathbb{P}} p^{v_p(n)}$ can be written as

$$\prod_{p\in\mathbb{P}} p^{u_p(n)},$$

where $u_p(n)$ runs over the (one or two) values 0 and $v_p(n)$ for all primes p. The number of the unitary divisors of n is $2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime divisors of n with $\omega(1) = 0$.

The concept of a unitary divisor (or a "block factor") originates from Vaidyanathaswamy [22] and was further studied for example by Cohen [4] and Tóth [21]. General accounts can be found, e.g., in [14, 18].

We denote the greatest common unitary divisor (gcud) of m and n as $(m,n)_{\oplus\oplus}$. The gcud of m and n exists for all $m, n \in \mathbb{Z}_+$ and

$$(m,n)_{\oplus\oplus} = \prod_{p\in\mathbb{P}} p^{\rho(v_p(m),v_p(n))},$$

where $\rho(v_p(m), v_p(n)) = v_p(m)$ if $v_p(m) = v_p(n)$, and $\rho(v_p(m), v_p(n)) = 0$ if $v_p(m) \neq v_p(n)$. We denote the least common unitary multiple (lcum) of m and n as $[m,n]_{\oplus\oplus}$. The lcum of m and n exists if and only if $v_p(m) = v_p(n)$, $v_p(n) = 0$ or $v_p(m) = 0$ for each prime p. For example, the lcum of 2 and 4 does not exist and the lcum of $12 = 2^2 3$ and $20 = 2^2 5$ exists and is equal to $60 = 2^2 \cdot 3 \cdot 5$. Hansen and Swanson [7] overcame the difficulty of the nonexistence of the lcum by defining

$$[m,n]^* = \frac{mn}{(m,n)_{\oplus\oplus}}.$$

It is easy to see that $[m,n]^*$ exists for all $m, n \in \mathbb{Z}_+$ and $[m,n]^* = [m,n]_{\oplus\oplus}$ when $[m,n]_{\oplus\oplus}$ exists. Naturally, $[m,n]^* \neq [m,n]_{\oplus\oplus}$ when $[m,n]_{\oplus\oplus}$ does not exist. For example, $[2,4]^* = 8$ but $[2,4]_{\oplus\oplus}$ does not exist. It is not reasonable to say that 8 is the lcum of 2 and 4, since $2, 4 \nparallel 8$.

It is easy to see that the unitary divisibility relation is a partial ordering on $\mathbb{Z}_+$. The gcud operation serves as the meet on this poset. Thus $\mathbb{Z}_+$ is a meet semilattice under the unitary divisibility relation. Unfortunately, however, it is not a lattice, since the lcum does not always exist. Korkee [12] embedded the unitary divisor meet semilattice $(\mathbb{Z}_+, \|)$ in a lattice by adding an element, denoted as ∞, so that each $n \in \mathbb{Z}_+$ is a unitary divisor of ∞. Then

$$m \vee n = \begin{cases} [m,n]_{\oplus\oplus} & \text{if } [m,n]_{\oplus\oplus} \text{ exists,} \\ \infty & \text{otherwise.} \end{cases}$$

Thus $(\mathbb{Z}_+ \cup \{\infty\}, \|)$ is a lattice. For example, the join of 2 and 3 is 6, and the join of 2 and 4 is ∞. This approach may be considered the one-point compactification of the discrete topological space $\mathbb{Z}_+$, [6].

The authors in [9] propose another way to embed the unitary divisor meet semilattice $(\mathbb{Z}_+, \|)$ in a lattice. They apply the one-point compactification with respect to each set of prime powers $\{p^a : a = 0, 1, 2, \ldots\}$ and then apply the fundamental theorem of arithmetic. This lattice may be considered a refinement of the lattice adopted by Korkee [12]. This extension goes as follows.

Let p be a prime and denote $U_p = \{p^a : a = 0, 1, 2, \ldots\}$. Let (U_p, τ) denote the corresponding discrete topological space. Let (U_p^*, τ^*) be the one-point compactification [6] of (U_p, τ). Denote the inserted point (often called the point of infinity) by p^∞. Thus $U_p^* = \{p^a : a = 0, 1, 2, \ldots \text{ or } a = \infty\} = \{1, p, p^2, \ldots, p^\infty\}$, where $1 \| p^a \| p^\infty$ for all $a = 0, 1, 2, \ldots, \infty$ and $p^a \nparallel p^b$ for all $0 < a, b < \infty$ with $a \neq b$. Then $(U_p^*, \|)$ is a lattice. In this lattice, for example, $1 \vee p = p$, $p \vee p^2 = p^\infty$, $p^2 \vee p^2 = p^2$, $p^2 \vee p^\infty = p^\infty$ for all $p \in \mathbb{P}$.

The lattice $(\mathbb{Z}_+^*, \|)$ is the direct product

$$\mathbb{Z}_+^* = \prod_{p \in \mathbb{P}} U_p^*$$

of the lattices $(U_p^*, \|)$. In [9], $\mathbb{Z}_+^*$ is defined topologically; we do not present these details here.

As an illustration of the lattice $(U_p^*, \|) = (\{1, p, p^2, \ldots, p^\infty\}, \|)$ its sublattice $(\{1, p, p^2, p^3, p^\infty\}, \|)$ is shown in Figure 1.

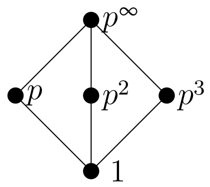

Figure 1. The sublattice $(\{1, p, p^2, p^3, p^\infty\}, \|)$ of the lattice $(U_p^*, \|)$

It should be noted that neither of the above extensions $(\mathbb{Z}_+ \cup \{\infty\}, \|)$ and $(\mathbb{Z}_+^*, \|)$ of $(\mathbb{Z}_+, \|)$ to a lattice is distributive nor locally finite. The main purpose of this paper is to emped $(\mathbb{Z}_+, \|)$ to a locally finite distributive lattice, see Section 2. As an application of distributivity, in Section 3 we consider semimultiplicative type functions, which we in turn utilize in meet and join type matrices in Section 4. Locally finity is needed to consider the Möbius function of the lattice, see Section 5.

2 Embedding to a locally finite distributive lattice

Consider first the meet semilattice $(\{1, p, p^2, p^3\}, \|)$, where p is a prime number. Its Hasse diagram is given in Figure 2.

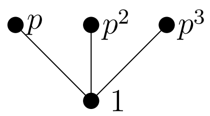

Figure 2. $(\{1, p, p^2, p^3\}, \|)$

We extend this to a distributive lattice as is shown in Figure 3. We denote this distributive extension as $(\{1, p, p^2, p^3\}^d, \|)$, where d stands for distributive.

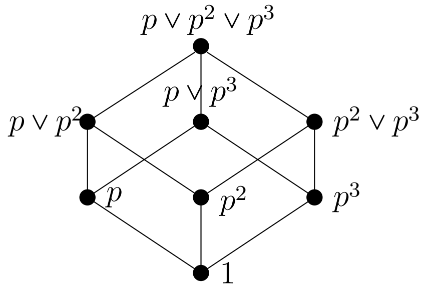

Figure 3. The distributive lattice $(\{1, p, p^2, p^3\}^d, \|)$

Let $\mathcal{P}(S)$ denote the power set of S. It is easy to see that

$$(\{1, p, p^2, p^3\}^d, \|) \simeq (\mathcal{P}(\{1,2,3\}), \subseteq).$$

We proceed in a similar manner to obtain

$$(\{1, p, p^2, \dots, p^n\}^d, \|) \simeq (\mathcal{P}(\{1,2,\dots,n\}), \subseteq).$$

Further, define $(\{1, p, p^2, \dots\}^d, \|)$ by

$$(\{1, p, p^2, \dots\}^d, \|) \simeq (\mathcal{P}'(\mathbb{Z}_+), \subseteq), \tag{1}$$

where $\mathcal{P}'(\mathbb{Z}_+)$ is the set of all *finite* subsets of $\mathbb{Z}_+$.

Each element $n_p \in \{1, p, p^2, \dots\}^d$ $(n \neq 1)$ is of the form

$$n_p = p^{i_1} \vee \dots \vee p^{i_k}, \quad k \geq 1, \tag{2}$$

where $i_1, \dots, i_k \in \mathbb{Z}_+$ are distinct. In other words, for each element $n_p \in \{1, p, p^2, \dots\}^d$ $(n \neq 1)$ there is a unique *finite* set $I_{n_p} \subseteq \mathbb{Z}_+$ such that

$$n_p = \bigvee_{i \in I_{n_p}} p^i. \tag{3}$$

In addition, we set $I_{n_p} = \varnothing$ for $n_p = 1$.

For $m_p, n_p \in \{1, p, p^2, \dots\}^d$ we have

$$m_p \| n_p \iff I_{m_p} \subseteq I_{n_p}. \tag{4}$$

The meet of $m_p, n_p \in \{1, p, p^2, \dots\}^d$ is given as

$$m_p \wedge n_p = \bigvee_{i \in I_{m_p} \cap I_{n_p}} p^i \tag{5}$$

and their join is

$$m_p \vee n_p = \bigvee_{i \in I_{m_p} \cup I_{n_p}} p^i. \tag{6}$$

The expression of n_p is the direct join; in symbols,

$$n_p = p^{i_1} \dot\vee \dots \dot\vee p^{i_k} = \dot\bigvee_{i \in I_{n_p}} p^i. \tag{7}$$

This means that n_p is the join of $A = \{p^{i_1}, \ldots, p^{i_k}\}$ and the set A is independent (that is, for each $p^{i_j} \in A$, $p^{i_j} \wedge \bigvee(A \setminus \{p^{i_j}\}) = 1$ (the least element of the lattice). For direct join, see [5].

Finally, we define $(\mathbb{Z}_+^d, \|)$ as a "restricted" direct product (or a "discrete" direct product [20]) by

$$(\mathbb{Z}_+^d, \|) = \prod_{p \in \mathbb{P}}^{*} (\{1, p, p^2, \ldots\}^d, \|). \tag{8}$$

This means that each element $n \in \mathbb{Z}_+^d$ is of the form

$$n = (n_2, n_3, \ldots, n_p, \ldots), \tag{9}$$

where $n_p \in \{1, p, p^2, \ldots\}^d$ for all primes $p \in \mathbb{P}$ and $n_p \neq 1$ for at most finitely many $p \in \mathbb{P}$. The word "restricted" stands for "$n_p \neq 1$ for at most finitely many $p \in \mathbb{P}$". This is denoted in (8) by the asterisk in $\prod^*$.

In other words, we have

$$(\mathbb{Z}_+^d, \|) \simeq \bigtimes_{i \in \mathbb{Z}_+}^{*} (\mathcal{A}_i, \subseteq), \tag{10}$$

where $\mathcal{A}_i = \mathcal{P}'(\mathbb{Z}_+)$ for all $i \in \mathbb{Z}_+$. This is again a "restricted" direct product, denoted by $\times^*$, which means that for each $(A_1, A_2, \ldots)$ in this restricted direct product, $A_i \in \mathcal{A}_i$ is nonempty for at most finitely many $i \in \mathbb{Z}_+$.

For $m, n \in \mathbb{Z}_+^d$ we have

$$m \| n \iff m_p \| n_p \ \ \forall p \in \mathbb{P}. \tag{11}$$

The meet and join of $m, n \in \mathbb{Z}_+^d$ are given as

$$m \wedge n = (m_2 \wedge n_2, m_3 \wedge n_3, \ldots, m_p \wedge n_p, \ldots) \tag{12}$$

and

$$m \vee n = (m_2 \vee n_2, m_3 \vee n_3, \ldots, m_p \vee n_p, \ldots) \tag{13}$$

For $n \in \mathbb{Z}_+^d$ let P_n denote the set of primes p for which $n_p \neq 1$. It is easy to see that if $m \| n$, then $P_m \subseteq P_n$, and if $P_m \cap P_n = \varnothing$, then $m \wedge n = 1$. The converse results do not hold.

For $m, n \in \mathbb{Z}_+$ we have $m \wedge n = (m, n)_{\oplus\oplus}$, and for $m, n \in \mathbb{Z}_+$ possessing the lcum we have $m \vee n = [m, n]_{\oplus\oplus}$. We here mean that for $n \in \mathbb{Z}_+$,

$$n = \prod_{p \in \mathbb{P}} p^{v_p(n)} = (2^{v_2(n)}, 3^{v_3(n)}, \ldots, p^{v_p(n)}, \ldots) = (p^{v_p(n)} : p \in \mathbb{P}). \tag{14}$$

Theorem 2.1. *The pair $(\mathbb{Z}_+^d, \|)$ is a locally finite distributive lattice.*

Proof. It is easy to see that $(\mathcal{P}'(\mathbb{Z}_+), \subseteq)$ is a distributive lattice; thus $(\{1, p, p^2, \ldots\}^d, \|)$ is a distributive lattice. Then $(\mathbb{Z}_+^d, \|)$ is distributive lattice as a restricted direct product of distributive lattices.

Since $(\mathcal{P}'(\mathbb{Z}_+), \subseteq)$ concerns finite sets, it is locally finite. We show that $(\mathbb{Z}_+^d, \|)$ is locally finite. Let $m, n \in \mathbb{Z}_+^d$ with $m \| n$, and consider the interval $[m, n] \subseteq \mathbb{Z}_+^d$. Then

$$m = (m_2, m_3, \ldots, m_p, \ldots), \quad n = (n_2, n_3, \ldots, n_p, \ldots),$$

where $m_p \| n_p$ for all primes $p \in \mathbb{P}$ and $m_p, n_p \neq 1$ for at most finitely many $p \in \mathbb{P}$. Since $(\{1, p, p^2, \ldots\}^d, \|)$ is locally finite, the number of elements in each interval $[m_p, n_p]$ is finite, and since $m_p, n_p \neq 1$ for at most finitely many $p \in \mathbb{P}$, the number of elements in $[m_p, n_p]$ is greater than 1 for at most finitely many $p \in \mathbb{P}$. Thus the number of elements in the interval $[m, n] \subseteq \mathbb{Z}_+^d$ is finite. This completes the proof. □

Recall that $(\mathbb{Z}_+^d, \|)$ is presented as the *restricted* direct product of the lattice of all *finite* subsets of $\mathbb{Z}_+$. We could also consider the *unrestricted* [3] (or *complete* [20]) direct product of the lattice of *all* subsets of $\mathbb{Z}_+$ as a distributive extension of the meet semilattice $(\mathbb{Z}_+, \|)$. This extension is even a Boolean lattice but not locally finite.

It should be noted that the unitary divisibility in $\mathbb{Z}_+^d$ is the divisibility relation with respect to multiplication $m \odot n$ in $\mathbb{Z}_+^d$ defined by $m \odot n = m \vee n$. Now, $(\mathbb{Z}_+^d, \odot)$ is a commutative semigroup of idempotents with identity, and we have $m \| n$ if and only if there exists $k \in \mathbb{Z}_+^d$ such that $n = m \odot k$. In particular, for $m, n \in \mathbb{Z}_+$, we have $m \| n$ if and only if there exists $k \in \mathbb{Z}_+$ such that $n = m \odot k$.

3 An analog of semimultiplicative functions

As an application of distributivity we introduce an analog of semimultiplicative functions. Distributivity is utilized in the proof of Theorem 3.1.

An arithmetical function $f : \mathbb{Z}_+ \to \mathbb{C}$ is said to be semimultiplicative if $f((m, n))f([m, n]) = f(m)f(n)$ for all $m, n \in \mathbb{Z}_+$, where (m, n) and $[m, n]$ stand for the gcd and lcm of $m, n \in \mathbb{Z}_+$. For information on semimultiplicative arithmetical functions we refer to [8, 15, 17, 18]

We propose an analog of this concept as follows. Let $(m, n)_d$ and $[m, n]_d$ denote the meet and join in $(\mathbb{Z}_+^d, \|)$. We say that a function $f : \mathbb{Z}_+^d \to \mathbb{C}$ is d-unitarily semimultiplicative if

$$f((m, n)_d)f([m, n]_d) = f(m)f(n), \ \forall m, n \in \mathbb{Z}_+^d. \tag{15}$$

Theorem 3.1. *Let f be a d-unitarily semimultiplicative function with $f(1) \neq 0$. Then*

$$f(p^{i_1} \vee \cdots \vee p^{i_k}) = f(1)^{1-k} f(p^{i_1}) \cdots f(p^{i_k}) \tag{16}$$

for all primes p and distinct positive integers $i_1, \ldots, i_k$ $(k \geq 1)$, and

$$f(m \vee n) = f(1)^{-1} f(m)f(n) \tag{17}$$

for all $m, n \in \mathbb{Z}_+^d$ with $P_m \cap P_n = \varnothing$. (Thus f is completely determined by its values at prime powers p^a, where $p \in \mathbb{P}$ and $a = 0, 1, 2, \ldots$)

Proof. We proceed by induction on k to prove (16). It is trivial for $k = 1$. Assume that (16) holds for some $k \geq 1$. We show that it holds for $k + 1$. Now,

$$f(p^{i_1} \vee \cdots \vee p^{i_k})f(p^{i_{k+1}}) = f((p^{i_1} \vee \cdots \vee p^{i_k}) \wedge p^{i_{k+1}})f((p^{i_1} \vee \cdots \vee p^{i_k}) \vee p^{i_{k+1}}).$$

By distributivity,

$$(p^{i_1} \vee \cdots \vee p^{i_k}) \wedge p^{i_{k+1}} = (p^{i_1} \wedge p^{i_{k+1}}) \vee \cdots \vee (p^{i_k} \wedge p^{i_{k+1}}) = 1 \vee \cdots \vee 1 = 1.$$

Thus

$$\begin{aligned} f(p^{i_1} \vee \cdots \vee p^{i_k} \vee p^{i_{k+1}}) &= f(1)^{-1} f(p^{i_1} \vee \cdots \vee p^{i_k}) f(p^{i_{k+1}}) \\ &= f(1)^{-1} f(1)^{1-k} f(p^{i_1}) \cdots f(p^{i_k}) f(p^{i_{k+1}}) \\ &= f(1)^{1-(k+1)} f(p^{i_1}) \cdots f(p^{i_k}) f(p^{i_{k+1}}). \end{aligned}$$

This proves (16).

If $P_m \cap P_n = \varnothing$, then $m \wedge n = (m,n)_d = 1$. We thus obtain (17). This completes the proof. □

4 Matrices associated with gcud and lcum

In this section we apply the new analog of semimultiplicative functions to a certain meet and join type matrix. For information on these type matrices, see, e.g., [2, 10, 11, 13].

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers with $x_1 < x_2 < \cdots < x_n$, and let f be an arithmetical function such that $f(x) \neq 0$ for all $x \in \mathbb{Z}_+$. Consider the $n \times n$ matrix $(S^\times)_f$ having

$$\frac{f^2((x_i, x_j)_{\oplus\oplus})}{f(x_i)(x_j)} \tag{18}$$

as its ij entry. We adopt the notation

$$B_f^*(x_i) = \sum_{\substack{d \| x_i \\ d \nparallel x_t \\ t<i}} g_f(d)$$

for all $i = 1, 2, \ldots, n$, where

$$g_f(x) = \sum_{d \| x} f^2\left(\frac{x}{d}\right) \mu^*(d).$$

The function μ^* is the unitary analog of the number-theoretic Möbius function [4].

Let $\overline{S} = \{d_1, d_2, \ldots, d_m\}$ be the minimal gcud-closed set containing S. Define the $n \times m$ matrix $C = (c_{ij})$ by

$$c_{ij} = \begin{cases} \dfrac{\sqrt{B_f^*(d_j)}}{f(x_i)} & \text{if } d_j \parallel x_i \\ 0 & \text{otherwise.} \end{cases}$$

It is shown in [9] that

$$(S^\times)_f = CC^T. \tag{19}$$

It is also shown in [9] that if S is gcud-closed, then

$$\det(S^\times)_f = \prod_{k=1}^{n} \frac{B_f^*(x_k)}{f^2(x_k)}. \tag{20}$$

If f is a d-unitarily semimultiplicative function on $\mathbb{Z}_+^d$ with $f(n) \neq 0$ for all $n \in \mathbb{Z}_+^d$, then

$$\frac{f^2((x_i, x_j)_{\oplus\oplus})}{f(x_i) f(x_j)} = \frac{f((x_i, x_j)_d)}{f([x_i, x_j]_d)}, \tag{21}$$

that is, the ij element of $(S^\times)_f$ is equal to $\frac{f((x_i,x_j)_d)}{f([x_i,x_j]_d)}$.

5 Möbius function

Let $(P, \leq)$ be a locally finite poset. The incidence algebra $I(P, \leq)$ of $(P, \leq)$ is the set of functions $f : P \times P \to \mathbb{C}$, which satisfy $f(x, y) = 0$ unless $x \leq y$. Addition and scalar multiplication are defined pointwise, and the multiplication or convolution is defined as

$$(f \star g)(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y).$$

The function $\delta \in I(P, \leq)$ is defined by $\delta(x, y) = 1$ if $x = y \in P$ and $\delta(x, y) = 0$ otherwise. The function δ serves as the identity under the convolution. The function $\zeta \in I(P, \leq)$ is defined by $\zeta(x, y) = 1$ if $x \leq y$. The Möbius function μ of $(P, \leq)$ is the inverse of the function ζ under the convolution. This means that

$$(\zeta \star \mu)(x, y) = \sum_{x \leq z \leq y} \mu(z, y) = \delta(x, y).$$

Material on incidence algebras and Möbius functions can be found in [1, 14, 16, 19].

Consider the lattices $(\{1, p, p^2, \dots\}^d, \|)$ and $(\mathbb{Z}_+^d, \|)$. According to Theorem 2.1 they are locally finite, and since they are lattices, they are posets. Therefore we can consider their Möbius functions for which we present expressions in the following theorem.

Theorem 5.1. *The Möbius function of the lattice* $(\{1, p, p^2, \dots\}^d, \|)$ *is given as*

$$\mu_p(m_p, n_p) = (-1)^{|I_{n_p}| - |I_{m_p}|} \text{ for } m_p \| n_p. \tag{22}$$

The Möbius function of the lattice $(\mathbb{Z}_+^d, \|)$ *is given as*

$$\mu_{\mathbb{Z}_+^d}(m, n) = \prod_{p \in P_m \cup P_n} \mu_p(m_p, n_p), \tag{23}$$

where P_k *is the set of primes* p *for which* $k_p \neq 1$ *(as is given in Section 2).*

Proof. We prove (22) by showing that the function $\mu_p(m_p, n_p)$ defined in (22) satisfies

$$\sum_{m_p \| d_p \| n_p} \mu_p(d_p, n_p) = \delta(m_p, n_p),$$

for $m_p, n_p \in \{1, p, p^2, \dots\}^d$, where $\delta(m_p, n_p)$ is the delta-function of $(\{1, p, p^2, \dots\}^d, \|)$. In fact,

$$\sum_{m_p \| d_p \| n_p} \mu_p(d_p, n_p) = \sum_{I_{m_p} \subseteq I \subseteq I_{n_p}} \mu_p(\bigvee_{i \in I} p^i, \bigvee_{i \in I_{n_p}} p^i) = \sum_{I_{m_p} \subseteq I \subseteq I_{n_p}} (-1)^{|I_{n_p}| - |I|}.$$

Let $|I_{m_p}| = r$ and $|I_{n_p}| = s$. Since $I_{m_p} \subseteq I_{n_p}$, we have $|I_{n_p} \setminus I_{m_p}| = s - r$. Thus we can write

$$\sum_{m_p \| d_p \| n_p} \mu_p(d_p, n_p) = \sum_{I \subseteq (I_{n_p} \setminus I_{m_p})} (-1)^{s - r - |I|} = \sum_{k=0}^{s-r} \binom{s-r}{k} (-1)^{s-r-|I|} = (1 + (-1))^{s-r}.$$

This is equal to 1 if $s = r$ and 0 if $s \neq r$. Here $s = r$ if and only if $m_p = n_p$. Thus

$$\sum_{m_p \| d_p \| n_p} \mu_p(d_p, n_p) = \delta(m_p, n_p),$$

This shows that the function μ_p defined in (22) is the Möbius function of $(\{1,p,p^2,\dots\}^d,\|)$.

We next prove (23). Assume first that $m \nparallel n$. Then

$$\mu_{\mathbb{Z}_+^d}(m,n) = 0.$$

On the other hand, $m_{p_0} \nparallel n_{p_0}$ for some prime p_0. Then $\mu_p(m_{p_0},n_{p_0}) = 0$. Now, $p_0 \in P_m \cup P_n$ (since otherwise $m_{p_0} = n_{p_0} = 1$ and $\mu_p(m_{p_0},n_{p_0}) = 1$), and therefore

$$\prod_{p\in P_m\cup P_n} \mu_p(m_p,n_p) = 0.$$

Thus, in this case,

$$\mu_{\mathbb{Z}_+^d}(m,n) = \prod_{p\in P_m\cup P_n} \mu_p(m_p,n_p).$$

Assume second that $m \| n$. Denote

$$S(m,n) = \sum_{m\|d\|n} \prod_{p\in P_d\cup P_n} \mu_p(d_p,n_p).$$

We show that $S(m,n) = \delta(m,n)$.

Now, in the above sum, $P_m \subseteq P_d \subseteq P_n$. Let $P_n = \{p_1,p_2,\dots,p_u\}$. Since $\mu_p(d_p,n_p) = 1$ for $p \notin \{p_1,p_2,\dots,p_u\}$, we have

$$S(m,n) = \sum_{m_{p_1}\|d_{p_1}\|n_{p_1}} \cdots \sum_{m_{p_u}\|d_{p_u}\|n_{p_u}} \mu_{p_1}(d_{p_1},n_{p_1})\cdots\mu_{p_u}(d_{p_u},n_{p_u}).$$

Thus

$$S(m,n) = \prod_{i=1}^{u} \sum_{m_{p_i}\|d_{p_i}\|n_{p_i}} \mu_{p_i}(d_{p_i},n_{p_i}) = \prod_{i=1}^{u} \delta_{p_i}(m_{p_i},n_{p_i}) = \prod_{p\in P_d\cup P_n} \delta_p(m_p,n_p).$$

If $p \notin P_d \cup P_n$, then $m_p = n_p = 1$ and therefore $\delta_p(m_p,n_p) = 1$. Thus

$$S(m,n) = \prod_{p\in\mathbb{P}} \delta_p(m_p,n_p).$$

By (9),

$$\prod_{p\in\mathbb{P}} \delta_p(m_p,n_p) = \delta(m,n).$$

This completes the proof. □

Remark 5.1. The unitary analog μ^* of the number-theoretic Möbius function is given as

$$\mu^*(n) = (-1)^{\omega(n)}, \quad n \in \mathbb{Z}_+,$$

where $\omega(n)$ is the number of distinct prime factors of n with $\omega(1) = 0$, see [4]. It is easy to see that $\mu_p(1,p^i) = -1 = \mu^*(p^i)$ for all prime powers p^i, and further, $\mu_{\mathbb{Z}_+^d}(1,n) = \mu^*(n)$ for all $n \in \mathbb{Z}_+$.

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