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On integers that are uniquely representable by modified arithmetic progressions

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Abstract: For positive integers $a, d, h, k, \gcd(a, d) = 1$, let $A = \{a, ha+d, ha+2d, \ldots, ha+kd\}$. We characterize the set of nonnegative integers that are uniquely representable by nonnegative integer linear combinations of elements of A.
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1 Introduction

Given integers $a_1, \ldots, a_k$, the linear equation $a_1x_1 + \cdots + a_kx_k = n$ has solutions in integers $x_1, \ldots, x_k$ if and only if $g = \gcd(a_1, \ldots, a_k)$ divides n. However, if we restrict our solutions to nonnegative integers, the divisibility condition is only necessary. It is therefore nontrivial to ask for the number of solutions $(x_1, \ldots, x_k)$ to the equation $a_1x_1 + \cdots + a_kx_k = n$, where each x_i is nonnegative, and without loss of generality to assume that $\gcd(a_1, \ldots, a_k) = 1$. The *Frobenius number* for the set $\{a_1, \ldots, a_k\}$ is the largest integer n for which this equations admits no solution in nonnegative integers $x_1, \ldots, x_k$. In an attempt to generalize the concept of Frobenius number, there has been some recent study (see [1, 3]) of determining the number of solutions and characterizing n which have exactly m solutions, for each nonnegative integer m. We have been unable to find any other references to this problem in the literature.

Given a positive and relative prime set of integers A and a nonnegative integer n, let $\mathtt{N}_A(n)$ denote the number of representations of n over nonnegative integers with elements in A. So if $A = \{a_1, \ldots, a_k\}$, with $\gcd(a_1, \ldots, a_k) = 1$, let

$$\mathtt{N}_A(n) = \left|\left\{(x_1, \ldots, x_k) : a_1x_1 + \cdots + a_kx_k = n, x_i \geq 0 \text{ for } 1 \leq i \leq k\right\}\right| \tag{1}$$

For each nonnegative integer m, let

$$\mathtt{S}_m(A) = \left\{n \in \mathbb{N} \cup \{0\} : \mathtt{N}_A(n) = m\right\}, \quad \mathtt{g}_m(A) = \max \mathtt{S}_m(A), \quad \mathtt{n}_m(A) = \left|\mathtt{S}_m(A)\right| \tag{2}$$

For positive integers a, d, h, k, $\gcd(a, d) = 1$, let $A = \{a, ha + d, ha + 2d, \ldots, ha + kd\}$. In this article, we determine the set $\mathtt{S}_1(A)$ of nonnegative integers that are uniquely expressible as a nonnegative integer linear combination of integers in A. In particular, we determine the largest integer and the number of integers in $\mathtt{S}_1(A)$. We remark that when $h = 1$, the integers in A are in arithmetic progression. In Section 2, we resolve the case $|A| = 2$. In Section 3, we solve the problem for $|A| > 2$.

2 Results for $A = \{a, b\}$

Let $A = \{a, b\}$, with $\gcd(a, b) = 1$. Explicit formula for $\mathtt{N}_A(n)$, given by Popoviciu [4], and later in a slightly different form by Tripathi [5], easily resolve the problem of determining $\mathtt{S}_m(A)$, $\mathtt{g}_m(A)$ and $\mathtt{n}_m(A)$, given in (2). Popoviciu showed that

$$\mathtt{N}_A(n) = \frac{n}{ab} - \left\{\frac{b^{-1}n}{a}\right\} - \left\{\frac{a^{-1}n}{b}\right\} + 1,$$

where $a^{-1}a \equiv 1 \pmod{b}$, $b^{-1}b \equiv 1 \pmod{a}$, and $\{x\} = x - \lfloor x \rfloor$ for $x \in \mathbb{R}$. We use a slightly different form of this formula, together with some properties satisfied by the function $\mathtt{N}_A(n)$ to resolve the problem.

Proposition 1. (Tripathi, [5]) *Let $A = \{a, b\}$, where a, b are positive and relatively prime. Then*

$$\mathtt{N}_A(n) = \frac{n + aa'(n) + bb'(n)}{ab} - 1,$$

where $a'(n) \equiv -na^{-1} \pmod{b}$, $1 \leq a'(n) \leq b$, and $b'(n) \equiv -nb^{-1} \pmod{a}$, $1 \leq b'(n) \leq a$.

The function $\mathtt{N}_A(n)$ satisfies

$$\mathtt{N}_A(n + kab) = \mathtt{N}_A(n) + k \textit{ for } k \geq 0.$$

For $0 \leq n \leq ab - 1$,

$$\mathtt{N}_A(n) = 0 \textit{ or } 1,$$

with

$$\mathtt{N}_A(n) = 1 \textit{ if and only if } n \in \Gamma(A).$$

In particular, $\mathtt{N}_A(n) = 1$ *for* $ab - a - b + 1 \leq n \leq ab - 1$. *Moreover,*

$$\mathtt{N}_A(m) + \mathtt{N}_A(n) = 1 \textit{ whenever } m + n = ab - a - b.$$

The problem of determining $\mathtt{S}_m(A)$, $\mathtt{g}_m(A)$ and $\mathtt{n}_m(A)$ was resolved by Beck & Robins [2], with perhaps a slightly different notation to describe their results. However, we provide a proof of this here not only for the sake of completeness towards resolution of the main problem, but also because it follows easily enough from Proposition 1.

Theorem 1. (Beck & Robins, [2]) *Let* $A = \{a, b\}$*, where* a, b *are positive and relatively prime. Then*

$$\mathtt{S}_0(A) = \mathtt{T}_0(A) = \big\{n : n = ab - aa'(n) - bb'(n)\big\},$$

and for $m \geq 1$,

$$\mathtt{S}_m(A) = \Big(\mathtt{T}_0(A) + mab\Big) \bigcup \Big(\mathtt{T}_1(A) + (m-1)ab\Big),$$

where

$$\mathtt{T}_i(A) = \mathtt{S}_i(A) \cap [0, ab-1] = \big\{n \in [0, ab-1] : n = (i+1)ab - aa'(n) - bb'(n)\big\}, i \in \{0, 1\}.$$

Moreover,

$$\max \mathtt{S}_m(A) = \max \mathtt{S}_0(A) + mab = (m+1)ab - a - b,$$

$$\big|\mathtt{S}_m(A)\big| = \big|\mathtt{S}_0(A)\big| = \tfrac{1}{2}(a-1)(b-1).$$

Proof: Let $\mathtt{T}_i(A) = \mathtt{S}_i(A) \cap [0, ab-1]$ for $i = 0, 1$. Note that $\mathtt{T}_0(A)$, $\mathtt{T}_1(A)$ partition $[0, ab-1]$. Hence, for each $t \geq 0$, $\mathtt{T}_0(A) + tab$, $\mathtt{T}_1(A) + tab$ partition $I_t = [tab, (t+1)ab - 1]$. Moreover, $\mathtt{N}_A(n) = t + i$ for $n \in \mathtt{T}_i(A) + tab$ for $i = 0, 1$. Hence $n \in \mathtt{S}_0(A) = \mathtt{T}_0(A)$ if and only if $0 = \mathtt{N}_A(n) = \frac{n + aa'(n) + bb'(n)}{ab} - 1$.

Let $m \geq 1$. Suppose $n \in \mathtt{S}_m(A)$. Then $n \in I_{m-1} \cup I_m$. If $n \in I_{m-1}$, then $n - (m-1)ab \in T_1$; if $n \in I_m$, then $n - mab \in T_0$. Therefore $n \in \big(\mathtt{T}_0(A) + mab\big) \cup \big(\mathtt{T}_1(A) + (m-1)ab\big)$.

Conversely, suppose $n \in \big(\mathtt{T}_0(A) + mab\big) \cup \big(\mathtt{T}_1(A) + (m-1)ab\big)$. If $n \in \mathtt{T}_0(A) + mab$, then $\mathtt{N}_A(n - mab) = 0$, so that $\mathtt{N}_A(n) = m$. If $n \in \mathtt{T}_1(A) + (m-1)ab$, then $\mathtt{N}_A\big(n - (m-1)ab\big) = 1$, so that $\mathtt{N}_A(n) = m$. Therefore $n \in \mathtt{S}_m(A)$.

Hence

$$\mathtt{S}_m(A) = \Big(\mathtt{T}_0(A) + mab\Big) \bigcup \Big(\mathtt{T}_1(A) + (m-1)ab\Big).$$

It follows that $\mathtt{S}_{t+1}(A) = \mathtt{S}_t(A) + ab$ for $t \geq 0$. Therefore

$$\max \mathtt{S}_m(A) = \max \mathtt{S}_0(A) + mab = (ab - a - b) + mab,$$

and

$$\big|\mathtt{S}_m(A)\big| = \big|\mathtt{S}_0(A)\big| = \tfrac{1}{2}(a-1)(b-1). \qquad \square$$

3 Results for modified arithmetic progressions

For positive integers a, d, h, k, $\gcd(a, d) = 1$, let $A = \{a, ha + d, \ldots, ha + kd\}$. The case $k = 1$ is completely resolved in Section 2. Throughout this section, we may therefore assume $k \geq 2$. Theorem 2, given below, resolves the problem of determining $\mathtt{S}_m(A)$ for $m = 0$.

Theorem 2. (Tripathi, [6]) *For positive integers* a, d, h, k*,* $\gcd(a, d) = 1$*, let* $A = \{a, ha + d, \ldots, ha + kd\}$*. Then*

$$\mathtt{S}_0(A) = \left\{ax + dy : 1 \leq y \leq a - 1, -\tfrac{dy}{a} < x \leq h \left\lceil \tfrac{y}{k} \right\rceil - 1\right\}.$$

In particular,

$$\begin{aligned} \max \mathtt{S}_0(A) &= ha \left\lfloor \tfrac{a-2}{k} \right\rfloor + (h-1)a + d(a-1); \\ \left|\mathtt{S}_0(A)\right| &= \tfrac{1}{2}h(a+r) \left\lceil \tfrac{a-1}{k} \right\rceil + \tfrac{1}{2}(a-1)(d-1), \end{aligned}$$

where $r \equiv a - 2 \pmod{k}$*,* $0 \leq r \leq k - 1$*.*

The main purpose of this section is to extend Theorem 2 to $m = 1$. For $m \geq 1$, we wish to determine the set of nonnegative integers n for which the equation

$$ax_0 + (ha+d)x_1 + (ha+2d)x_2 + \cdots + (ha+kd)x_k = a\left(x_0 + h\sum_{i=1}^{k} x_i\right) + d\left(\sum_{i=1}^{k} ix_i\right) = n \quad (3)$$

has exactly m solutions in nonnegative integer tuples $(x_0, x_1, \ldots, x_k)$. In the general case, this appears difficult to achieve, and we restrict ourselves mainly to the case $m = 1$. When $a = 1$, it is easy to see that the set $\mathtt{S}_1(A) = \{0, 1, 2, \ldots, h + d - 1\}$. Henceforth, throughout this section, we may therefore assume $a > 1$.

Each integer n can be expressed by the form $ax + dy$ with $x, y \in \mathbb{Z}$ since $\gcd(a, d) = 1$. Since each $x_i \geq 0$ in (3), n must be of the form $ax + dy$ with $x, y \geq 0$ in order that $\mathtt{N}_A(n)$ be nonzero. Thus $\mathtt{N}_A(n) = 0$ for $n < 0$. Suppose n in (3) is of the form $ax + dy$ with $x, y \geq 0$. Since $\gcd(a, d) = 1$, we must have

$$\sum_{i=1}^{k} ix_i \equiv y \pmod{a}.$$

The set $\mathtt{S}_1(A)$ consists of nonnegative integers that are 1-representable. If $\mathbf{m}_x$ denotes the least nonnegative integer n in the residue class x modulo a that is representable by A, then $\mathtt{S}_1(A)$ must contain exactly those $\mathbf{m}_x$ for which $\mathtt{N}_A(\mathbf{m}_x) = 1$. To determine $\mathbf{m}_x$, we must minimize $x_0 + h\sum_{i=1}^{k} x_i$ subject to $\sum_{i=1}^{k} ix_i \equiv x \pmod{a}$, where each $x_i \geq 0$. This is equivalent to minimizing $\sum_{i=1}^{k} x_i$ subject to the same constraint since $x_0 = 0$ for the minimum and since $h > 0$. Now suppose $\sum_{i=1}^{k} ix_i = qk + r$, where $0 \leq r \leq k - 1$. To minimize $\sum_{i=1}^{k} x_i$, we must choose $x_k = q$, and additionally $x_r = 1$ in case $r \neq 0$; in both cases, we choose all other $x_i = 0$. Thus the minimum value of $\sum_{i=1}^{k} x_i$ is q if $r = 0$ and $q + 1$ is $r \neq 0$. The two cases can be combined to yield the formula $\lceil \frac{s}{k} \rceil$, where $\sum_{i=1}^{k} ix_i = s$. We record this as

$$\min\left\{\sum_{i=1}^{k} x_i : \sum_{i=1}^{k} ix_i = s, x_i \geq 0\right\} = \left\lceil \frac{s}{k} \right\rceil \quad (4)$$

Since $\gcd(a,d) = 1$, the set $\{dy : 0 \le y \le a-1\}$ represents a complete residue system modulo a. Therefore, by Theorem 2,

$$\mathbf{m}_{dy} = ha\left\lceil \tfrac{y}{k} \right\rceil + dy \tag{5}$$

for $1 \le y \le a-1$. In fact, the same formula also applies to the case $y = 0$.

Proposition 2. *For positive integers a, d, h, k, $\gcd(a,d) = 1$, let $A = \{a, ha+d, \ldots, ha+kd\}$. Then*

$$\mathtt{N}_A\big(\mathbf{m}_{dy}\big) = \begin{cases} 1, & \text{if } y \le k \text{ or } k \mid y \text{ or } k \mid (y+1); \\ > 1, & \text{otherwise.} \end{cases}$$

In particular, if $k \ge a$, then $\mathbf{m}_{dy} = ha + dy$ and $\mathtt{N}_A\big(\mathbf{m}_{dy}\big) = 1$ for $1 \le y \le a-1$.

Proof: Suppose $(x_0, \ldots, x_k)$ is a solution to (3) with $n = \mathbf{m}_{dy} = ha\left\lceil \frac{y}{k} \right\rceil + dy$. Then $\sum_{i=1}^{k} ix_i \equiv y \pmod{a}$. If $\sum_{i=1}^{k} ix_i > y$, then $x_0 + h\sum_{i=1}^{k} x_i \ge h\lceil \frac{y+a}{k} \rceil$ by (4). But then the LHS in (3) is greater than $\mathbf{m}_{dy}$. Therefore $\sum_{i=1}^{k} ix_i = y$. We show there is a unique solution if $y \le k$ or $k \mid y$ or $k \mid (y+1)$, but not in any other case.

If $1 \le y \le k$, we must have $\sum_{i=1}^{k} ix_i = y$ and $x_0 + h\sum_{i=1}^{k} x_i = h$. By (4), the minimum value assumed by $\sum_{i=1}^{k} x_i$ is 1, and this is possible only when $x_y = 1$, with all other $x_i = 0$. Hence $\mathtt{N}_A\big(\mathbf{m}_{dy}\big) = 1$ in this case.

If $y = qk$ or $qk - 1$, with $q \ge 1$, we must have $\sum_{i=1}^{k} ix_i = y$ and $x_0 + h\sum_{i=1}^{k} x_i = hq$. By (4), the minimum value assumed by $\sum_{i=1}^{k} x_i$ is q. If $\sum_{i=1}^{k} x_i > q$, then $x_0 + h\sum_{i=1}^{k} x_i > hq$. So we must have $\sum_{i=1}^{k} ix_i = y$ and $\sum_{i=1}^{k} x_i = q$.

If $y = qk$, the only possibility is $x_k = q$, with all other $x_i = 0$. If $y = qk - 1$, the only possibility is $x_k = q-1$, $x_{k-1} = 1$, with all other $x_i = 0$. In all cases, $\mathtt{N}_A\big(\mathbf{m}_{dy}\big) = 1$.

We now show that $\mathtt{N}_A\big(\mathbf{m}_{dy}\big) > 1$ in all other cases. In all other cases, $y = qk+r$, $1 \le r \le k-2$ and $q \ge 1$. Note that there is no y in this case if $k = 2$, so we may henceforth also assume $k \ge 3$. It is easily verified that both (i) $x_k = q$, $x_r = 1$, and (ii) $x_k = q-1$, $x_{k-1} = 1$, $x_{r+1} = 1$ (set $x_{k-1} = 2$ if $r = k-2$) are solutions; in both cases, we set all other $x_i = 0$. This completes the proof that $\mathtt{N}_A\big(\mathbf{m}_{dy}\big) > 1$ for these cases. $\square$

Proposition 3. *For positive integers a, d, h, k, $\gcd(a,d) = 1$, let $A = \{a, ha+d, \ldots, ha+kd\}$. If $\mathtt{N}_A\big(n\big) \ge 1$, then*

$$\mathtt{N}_A\big(n + ma\big) > \mathtt{N}_A\big(n\big),$$

where $m = h\lceil \frac{a}{k} \rceil + d$.

Proof: Suppose $(x_0, x_1, \ldots, x_k)$ is a solution to (3) for n. Then $(x_0 + m, x_1, \ldots, x_k)$ is a solution to (3) for $n + ma$. To find a distinct solution, write $a = qk + r$, $0 \le r < k$. Set $x_k' = x_k + q$, $x_r' = x_r$ if $r = 0$ and $x_r' = x_r + 1$ if $r > 0$, with all other $x_i' = x_i$. Then $\sum_{i=1}^{k} x_i' - \sum_{i=1}^{k} x_i$ equals q if $r = 0$ and $q+1$ if $r > 1$; and $\sum_{i=1}^{k} ix_i' - \sum_{i=1}^{k} ix_i$ equals qk if $r = 0$ and $qk + r$ if $r > 1$. Hence

$$\sum_{i=1}^{k} x_i' - \sum_{i=1}^{k} x_i = \left\lceil \frac{a}{k} \right\rceil, \quad \sum_{i=1}^{k} ix_i' - \sum_{i=1}^{k} ix_i = a.$$

It follows that $(x_0', x_1', \ldots, x_k')$ is a solution to (3) for $n+ma$ distinct from $(x_0+m, x_1, \ldots, x_k)$. Thus $\mathtt{N}_A\big(n+ma\big) > \mathtt{N}_A\big(n\big)$ whenever $\mathtt{N}_A\big(n\big) \geq 1$. □

Proposition 4. *For positive integers* a, d, h, k, $\gcd(a,d) = 1$, *let* $A = \{a, ha+d, \ldots, ha+kd\}$. *If* $\mathtt{N}_A\big(n\big) \geq 1$ *for some* $n \equiv dy \pmod{a}$ *with* $1 < y < a$, *then*

$$\mathtt{N}_A\big(n+ha\big) > \mathtt{N}_A\big(n\big).$$

Proof: Suppose $(x_0, x_1, \ldots, x_k)$ is a solution to (3) for n. Then $(x_0+h, x_1, \ldots, x_k)$ is a solution to (3) for $n+ha$. To find a distinct solution, write $y = qk+r$. If $q = 0$, there is a solution with $x_r = 1$; then $(x_0, x_1+1, \ldots, x_{r-1}+1, x_r-1, \ldots, x_k)$ (merge x_1+1 and $x_{r-1}+1$ into $x_{r-1}+2$ when $r=2$) is a solution to (3) for $n+ha$. If $q > 0$, there is a solution with $x_k = q$; then $(x_0, x_1+1, \ldots, x_{k-1}+1, x_k-1)$ is a solution to (3) for $n+ha$. Since we obtain solutions distinct from $(x_0+h, x_1, \ldots, x_k)$ in both cases, we have $\mathtt{N}_A\big(n+ha\big) > \mathtt{N}_A\big(n\big)$ in all cases where $y \in \{2, 3, \ldots, a-1\}$. □

Proposition 5. *For positive integers* a, d, h, k, $\gcd(a,d) = 1$, *let* $A = \{a, ha+d, \ldots, ha+kd\}$. *If* ℓ *is the least positive integer such that* $\mathtt{N}_A\big(\mathbf{m}_{dy}+\ell a\big) > \mathtt{N}_A\big(\mathbf{m}_{dy}\big)$, *then*

$$\ell = \begin{cases} h\lceil \frac{a}{k} \rceil + d, & \textit{if } y = 0; \\ h\lfloor \frac{a}{k} \rfloor + d, & \textit{if } y = 1; \\ h, & \textit{if } 1 < y < a,\ h \leq d; \\ h, & \textit{if } 1 < y < a,\ h > d,\ k \leq a+1; \\ d, & \textit{if } 1 < y \leq k-a,\ h > d,\ a+1 < k < 2a-1; \\ h, & \textit{if } k-a < y < a,\ h > d,\ a+1 < k < 2a-1; \\ d, & \textit{if } 1 < y < a,\ h > d,\ k \geq 2a-1. \end{cases}$$

Proof: We use $m = h\lceil \frac{a}{k} \rceil + d$ in the first two cases of this proof.

CASE 1. $\big(y = 0\big)$

In view of Propositions 2 and 3, it is enough to show that $\mathtt{N}_A\big(\mathbf{m}_{dy}+(m-1)a\big) = \mathtt{N}_A\big(\mathbf{m}_{dy}\big)$, which reduces to showing $\mathtt{N}_A\big((m-1)a\big) = 1$.

Suppose $(x_0, \ldots, x_k)$ is a solution to (3) with $n = (m-1)a$. Then $\sum_{i=1}^k ix_i = ta$, with $t \geq 0$. If $\sum_{i=1}^k ix_i \geq a$, then $\sum_{i=1}^k x_i \geq \lceil \frac{a}{k} \rceil$ by (4). But then the LHS in (3) is at least as much as ma. So $\sum_{i=1}^k ix_i = 0$, which implies $x_i = 0$ for $i > 0$. This forces $x_0 = m-1$. Hence $\mathtt{N}_A\big((m-1)a\big) = 1$, as desired.

CASE 2. $\big(y = 1\big)$

Note that $\lceil \frac{a}{k} \rceil$ equals $\lfloor \frac{a}{k} \rfloor$ when $k \mid a$, and $1 + \lfloor \frac{a}{k} \rfloor$ when $k \nmid a$. We therefore consider two subcases.

Subcase (i). $\big(k \mid a\big)$

In view of Propositions 2 and 3, it is enough to show that $\mathtt{N}_A\big(\mathbf{m}_{dy}+(m-1)a\big) = \mathtt{N}_A\big(\mathbf{m}_{dy}\big)$, which reduces to showing $\mathtt{N}_A\big((h+m-1)a+d\big) = 1$.

Suppose $(x_0, \ldots, x_k)$ is a solution to (3) with $n = (h+m-1)a+d$. Then $\sum_{i=1}^k ix_i = ta+1$, with $t \geq 0$. If $\sum_{i=1}^k ix_i \geq a+1$, then $\sum_{i=1}^k x_i \geq \lceil \frac{a+1}{k} \rceil = \lceil \frac{a}{k} \rceil + 1$ by (4). But then the LHS in

(3) is at least as much as $(h+m)a+d$. So $\sum_{i=1}^k ix_i = 1$, which implies $x_1 = 1$, $x_i = 0$ for $i > 1$. This forces $x_0 = m-1$. Hence $\mathtt{N}_A\big((m-1)a\big) = 1$, as desired.

Subcase (ii). $\big(k \nmid a\big)$

In view of Propositions 2 and 3, it is enough to show that $\mathtt{N}_A\big(\mathbf{m}_{dy} + (m-h)a\big) > \mathtt{N}_A\big(\mathbf{m}_{dy} + (m-h-1)a\big) = \mathtt{N}_A\big(\mathbf{m}_{dy}\big)$. This amounts to showing $\mathtt{N}_A\big(ma+d\big) > \mathtt{N}_A\big((m-1)a+d\big) = 1$.

Suppose $(x_0,\dots,x_k)$ is a solution to (3) with $n = (m-1)a+d$. As in subcase (i), if $\sum_{i=1}^k ix_i \geq a+1$, then $\sum_{i=1}^k x_i \geq \lceil \frac{a+1}{k} \rceil = \lceil \frac{a}{k} \rceil$ by (4). Thus the LHS in (3) is at least as much as $ha\lceil \frac{a}{k} \rceil + d(a+1) > ma$. Hence $\sum_{i=1}^k ix_i = 1$, so that $\sum_{i=1}^k x_i = 1$, which forces $x_0 = m-h-1$. This shows that $\mathtt{N}_A\big((m-1)a+d\big) = 1$, as desired.

We now exhibit two solutions to (3) with $n = ma+d$. If $\sum_{i=1}^k ix_i = 1$, we get $x_1 = 1$ and $x_i = 0$ for $i > 1$. Thus $\sum_{i=1}^k x_i = 1$, and setting $x_0 = m-h$ provides a solution. For a second solution, suppose $\sum_{i=1}^k ix_i = a+1$. By (4), we may choose $x_1,\dots,x_k$ such that $\sum_{i=1}^k x_i = \lceil \frac{a+1}{k} \rceil = \lceil \frac{a}{k} \rceil$. Setting $x_0 = 0$ provides a solution. Hence $\mathtt{N}_A\big(ma+d\big) > 1$, as desired.

CASE 3. $\big(1 < y < a$, $h \leq d$, or $1 < y < a$, $h > d$, $k \leq a+1$, or $k-a < y < a$, $h > d$, $a+1 < k < 2a-1\big)$

In view of Proposition 4, it is enough to show that $\mathtt{N}_A\big(\mathbf{m}_{dy} + (h-1)a\big) = \mathtt{N}_A\big(\mathbf{m}_{dy}\big)$.

We claim that $(x_0, x_1, \dots, x_k)$ is a solution to (3) with $n = \mathbf{m}_{dy}$ if and only if $(x_0+h-1, x_1, \dots, x_k)$ is a solution to (3) with $n = \mathbf{m}_{dy} + (h-1)a$.

If $(x_0, x_1, \dots, x_k)$ is a solution to (3) with $n = \mathbf{m}_{dy}$, it is easily verified that $(x_0+h-1, x_1, \dots, x_k)$ is a solution to (3) with $n = \mathbf{m}_{dy} + (h-1)a$. Conversely, suppose $(x_0', x_1', \dots, x_k')$ is a solution to (3) with $n = \mathbf{m}_{dy} + (h-1)a$. Thus $\sum_{i=1}^k ix_i' \equiv y \pmod{a}$.

If $\sum_{i=1}^k ix_i' \geq y+a$, then $\sum_{i=1}^k x_i' \geq \lceil \frac{y+a}{k} \rceil$ by (4). Hence the LHS in (3) is at least as much as $ha\lceil \frac{y+a}{k} \rceil + d(y+a)$.

If $h \leq d$, then for $1 < y < a$, $ha\lceil \frac{y+a}{k} \rceil + d(y+a) \geq ha\lceil \frac{y}{k} \rceil + dy + ha = \mathbf{m}_{dy} + ha$.

If $h > d$ and $k \leq a$, then for $1 < y < a$, $ha\lceil \frac{y+a}{k} \rceil + d(y+a) > ha\big(1 + \lceil \frac{y}{k} \rceil\big) + dy = \mathbf{m}_{dy} + ha$.

If $h > d$ and $k = a+1$, then for $1 < y < a$, $\frac{y}{k} < 1 < \frac{y+a}{k} < 2$. Hence $ha\lceil \frac{y+a}{k} \rceil + d(y+a) > ha\big(1 + \lceil \frac{y}{k} \rceil\big) + dy = \mathbf{m}_{dy} + ha$.

If $h > d$ and $a+1 < k < 2a-1$, then for $k-a < y < a$, $\lceil \frac{y+a}{k} \rceil > \lceil \frac{y}{k} \rceil$ since $\frac{y+a}{k} > 1 > \frac{y}{k}$. Therefore $ha\lceil \frac{y+a}{k} \rceil + d(y+a) \geq ha\big(1 + \lceil \frac{y}{k} \rceil\big) + dy = \mathbf{m}_{dy} + ha$.

Hence $\sum_{i=1}^k ix_i' = y$. Therefore $(x_0', x_1', \dots, x_k')$ must satisfy $x_0' + h\sum_{i=1}^k x_i' = h\big(1 + \lceil \frac{y}{k} \rceil\big) - 1$. Hence $x_0' \equiv -1 \pmod{h}$, and since x_0' must be nonnegative, $x_0' \geq h-1$. It now follows that $\big(x_0' - (h-1), x_1', \dots, x_k'\big)$ is a solution to (3) with $n = \mathbf{m}_{dy}$.

CASE 4. $\big(1 < y \leq k-a$, $h > d$, $a+1 < k < 2a-1$, or $1 < y < a$, $h > d$, $k \geq 2a-1\big)$

Since $y < a < k$, we have $\mathbf{m}_{dy} = ha + dy$ by (5) and $\mathtt{N}_A\big(\mathbf{m}_{dy}\big) = 1$ by Proposition 2. Therefore we must show that $\mathtt{N}_A\big(ha + dy + da\big) > \mathtt{N}_A\big(ha + dy + (d-1)a\big) = 1$.

Suppose $(x_0,\dots,x_k)$ is a solution to (3) with $n = (h+d-1)a + dy$. Thus $\sum_{i=1}^k ix_i \equiv y \pmod{a}$. If $\sum_{i=1}^k ix_i \geq y+a$, then $\sum_{i=1}^k x_i \geq 1$ by (4). But then the LHS of (3) is at least as much as $ha + d(y+a)$, which is greater than $(h+d-1)a + dy$. Therefore $\sum_{i=1}^k ix_i = y$. If $\sum_{i=1}^k x_i \geq 2$, we must have $(h+d-1)a + dy \geq a(x_0 + 2h) + dy$. But then $x_0 \leq d-1-h < 0$. Thus $\sum_{i=1}^k x_i = 1$. Hence $a(x_0+h) + dy = (h+d-1)a + dy$, so that $x_0 = d-1$. Therefore $\mathtt{N}_A\big(ha + dy + (d-1)a\big) = 1$.

We now exhibit two solutions to (3) with $n = (h+d)a + dy$. For the first solution, we choose $x_1, \ldots, x_k$ such $\sum_{i=1}^{k} ix_i = y$ and $\sum_{i=1}^{k} x_i = 1$. Thus we must have $a(x_0 + h) + dy = ha + d(y+a)$, so that $x_0 = d$. For the second solution, choose $x_1, \ldots, x_k$ such $\sum_{i=1}^{k} ix_i = y + a$ and $\sum_{i=1}^{k} x_i = \lceil \frac{y+a}{k} \rceil = 1$. Thus we must have $a(x_0+h) + d(y+a) = ha + d(y+a)$, so $x_0 = 0$. Therefore $\mathtt{N}_A\big(ha + dy + da\big) \geq 2$.

This completes the proof. $\square$

Propositions 2, 3, 4, and 5 are each vital in providing a complete description of $\mathtt{S}_1(A)$, on lines similar to Theorem 2.

Theorem 3. *For positive integers a, d, k, h, $\gcd(a,d) = 1$, let $A = \{a, ha+d, ha+2d, \ldots, ha + kd\}$. Let*

$$
\begin{aligned}
T_1 &= \Big\{ta : 0 \leq t \leq h \left\lceil \tfrac{a}{k} \right\rceil + d - 1\Big\},\\
T_2 &= \Big\{ha + d + ta : 0 \leq t \leq h \left\lfloor \tfrac{a}{k} \right\rfloor + d - 1\Big\},\\
T_3 &= \Big\{ha + dy + ta : 1 < y < k - 1, 0 \leq t \leq h - 1\Big\},\\
T_4 &= \Big\{ha + dy + ta : 1 < y < a, 0 \leq t \leq d - 1\Big\},\\
T_5 &= \Big\{ha + dy + ta : 1 < y < a, 0 \leq t \leq h - 1\Big\},\\
T_6 &= \Big\{ha + dy + ta : 1 < y \leq k - a, 0 \leq t \leq d - 1\Big\},\\
T_7 &= \Big\{ha + dy + ta : k - a < y < a, 0 \leq t \leq h - 1\Big\},\\
T_8 &= \Big\{q(ha + dk) + ta : 1 \leq q \leq \left\lfloor \tfrac{a-1}{k} \right\rfloor, 0 \leq t \leq h - 1\Big\},\\
T_9 &= \Big\{q(ha + dk) - d + ta : 1 \leq q \leq \left\lfloor \tfrac{a}{k} \right\rfloor, 0 \leq t \leq h - 1\Big\}.
\end{aligned}
$$

(i) *If $k \leq a + 1$, then $\mathtt{S}_1(A) = T_1 \cup T_2 \cup T_3 \cup T_8 \cup T_9$.*

(ii) *If $h \leq d$ and $k > a + 1$, then $\mathtt{S}_1(A) = T_1 \cup T_2 \cup T_5$.*

(iii) *If $h > d$ and $a + 1 < k < 2a - 1$, then $\mathtt{S}_1(A) = T_1 \cup T_2 \cup T_6 \cup T_7$.*

(iv) *If $h > d$ and $k \geq 2a - 1$, then $\mathtt{S}_1(A) = T_1 \cup T_2 \cup T_4$.*

In particular,

$$
\max \mathtt{S}_1(A) = \begin{cases}
h\left(1 + \lfloor \frac{a}{k} \rfloor\right) + d - 1\big)a + d, & \text{if } k \leq a;\\
\max\left\{(h+d-1)a + d, (2h-1)a + d(a-1)\right\}, & \text{if } k = a+1;\\
\max\left\{(h+d-1)a + d, (2h-1)a + d(a-1)\right\}, & \text{if } h \leq d,\ k > a+1;\\
(2h-1)a + d(a-1), & \text{if } h > d,\ a+1 < k < 2a-1;\\
(h+d-1)a + d(a-1), & \text{if } h > d,\ k \geq 2a-1.
\end{cases}
$$

$$\left|\mathtt{S}_1(A)\right| = \begin{cases} h\left(2\left\lceil \frac{a}{k}\right\rceil + 2\left\lfloor \frac{a}{k}\right\rfloor + k - 4\right) + 2d, & \textit{if } k \le a+1; \\ h(a-1) + 2d, & \textit{if } h \le d,\ k > a+1; \\ h(2a-k) + d(k-a+1), & \textit{if } h > d,\ a+1 < k < 2a-1; \\ h + ad, & \textit{if } h > d,\ k \ge 2a-1. \end{cases}$$

Proof: For each $y \in \{0, 1, \ldots, a-1\}$, $\mathbf{m}_{dy}$ denotes the least nonnegative integer $n \equiv dy \pmod{a}$ for which $\mathtt{N}_A(n) > 0$. By Proposition 2, $\mathbf{m}_{dy} \in \mathtt{S}_1(A)$ if and only if $y \le k$ or $k \mid y$ or $k \mid (y+1)$. Proposition 5 determines the least positive integer ℓ for which $\mathtt{N}_A\big(\mathbf{m}_{dy} + \ell a\big) > \mathtt{N}_A\big(\mathbf{m}_{dy}\big)$. Since $\mathtt{N}_A\big(n+a\big) \ge \mathtt{N}_A\big(n\big)$ for each $n \ge 0$,

$$\mathtt{S}_1(A) = \big\{\mathbf{m}_{dy}, \mathbf{m}_{dy} + a, \ldots, \mathbf{m}_{dy} + (\ell-1)a : y \le k, \text{ or } k \mid y, \text{ or } k \mid (y+1)\big\}.$$

We consider the two cases (I) $h \le d$, and (II) $h > d$. Case (I) further has two subcases: (i) $k \le a+1$, and (ii) $k > a+1$. Case (II) has three subcases: (i) $k \le a+1$, (ii) $a+1 < k < 2a-1$, and (iii) $k \ge 2a-1$. Listing the elements in the set $\mathtt{S}_1(A)$ follows from a careful examination of the value of ℓ from Proposition 5. Computation of the largest element and the size of the set $\mathtt{S}_1(A)$ follows routinely from the characterization of $\mathtt{S}_1(A)$. $\square$

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