Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 22, 2016, No. 3, 20–35

Upper bound of embedding index in grid graphs

M. Kamal Kumar [1] and R. Murali [2]

[1] Department of the Mathematics, CMR Institute of Technology
Bangalore 560037, India
e-mail: kamalmvz@gmail.com

[2] Department of the Mathematics, Dr. Ambedkar Institute of Technology
Bangalore 560056, India
e-mail: muralir2968@gmail.com

Received: 3 August 2015 **Revised:** 30 September 2016 **Accepted:** 2 October 2016

Abstract: A subset S of the vertex set of a graph G is called a dominating set of G if each vertex of G is either in S or adjacent to at least one vertex in S. A partition $D = \{D_1, D_2, \ldots, D_k\}$ of the vertex set of G is said to be a domatic partition or simply a d-partition of G if each class D_i of D is a dominating set in G. The maximum cardinality taken over all d-partitions of G is called the domatic number of G denoted by $d(G)$. A graph G is said to be domatically critical or d-critical if for every edge x in G, $d(G–x) < d(G)$, otherwise G is said to be domatically non d-critical. The embedding index of a non d-critical graph G is defined to be the smallest order of a d-critical graph H containing G as an induced subgraph denoted by $\theta(G)$. In this paper, we find the upper bound of $\theta(G)$ for grid graphs.
Keywords: Domination number, Domatic partition, Domatic number, d-Critical graphs.
AMS Classification: 05C69.

1 Introduction

A subset S of the vertex set of a graph G is called a dominating set of G if each vertex of G is either in S or adjacent to at least one vertex in S. A partition $D = \{D_1, D_2, \ldots, D_k\}$ of the vertex set of G is said to be a domatic partition or simply a d-partition of G if each class D_i of D is a dominating set in G. The maximum cardinality taken over all d partitions of G is called the domatic number of G denoted by $d(G)$.

A graph G is said to be domatically critical or d-critical if for every edge x in G, $d(G–x) < d(G)$, otherwise G is said to be domatically non d-critical. This concept was introduced by E. J. Cockayne and S. T. Hedetniemi in [3]. Further study on this class of graphs was carried out by B. Zelinka [1], D. F. Rall [2], H. B. Walikar, A. P. Deshpande, Savita Basapur and L. Sudershan Reddy [7, 8, 9 10, 11].

It is not difficult to see that $d(G)-1 \le d(G-x) \le d(G)$ for any edge x in G and thus, the d-critical graphs may also be defined as those graphs G for which $d(G-x) = d(G)-1$ holds for every edge x in G. Graphs that are critical with respect to a given property frequently play an important role in the investigation of that property. The critical concept in graph theory was introduced by Dirac [6] in 1952 with respect to chromatic number of a graph mainly to study the four color conjecture. In [7, 8] it was proved that "Every non d-critical graph can be embedded in some d-critical graph" by constructing a d-critical graph of order $2p$ containing a given non d-critical graph of order p. Also, the path P_{3n-1}, a non d-critical graph can be viewed as an induced subgraph of the cycle C_{3n}, a d-critical graph. The embedding index of a non d-critical graph G is defined to be the smallest order of d-critical graph H containing G as an induced subgraph denoted by $\theta(G)$, i.e., $\theta(G) = \min \{p(H) - p(G) : H \in F\}$ where F is the family of all d-critical graphs H containing G as an induced subgraph (where $p(H)$ denotes the order of H).

It is obvious that $1 \le \theta(G) \le p(G)$ for any non d-critical graph G.

A graph is said to be domatically full if and only if $d(G) = \delta(G)+1$, where $\delta(G)$ denotes the minimum degree of G.

Throughout this paper, by a graph G we mean a finite, undirected graph without multiple edges or loops. By P_n we mean a path of n vertices.

2 Preliminary results

2.1 Theorem [1]. Let G be a domatically critical graph with domatic number $d(G) = d$. Then, the vertex set $V(G)$ of G is the union of d pairwise disjoint sets $V_1, V_2, \ldots, V_d$ with the property that for any two distinct integers $i, j : 1 \le i, j \le d$, the subgraph G_{ij} of G induced by the set $V_i \cup V_j$ is a bipartite graph on the sets V_i, V_j, all of whose connected components are stars.

2.2 Theorem [1]. A regular domatically full graph G with n vertices and with a domatic number d exists if and only if d divides n. Such a graph is also domatically critical.

2.3 Theorem [9]. If G is regular and domatically full, Then, G is domatically critical, however the converse is not true.

2.4 Theorem [8]. If G is a domatically full graph in which every edge is incident with a vertex of minimum degree, Then, G is d-critical.

2.1 Examples

Figure 1 below shows a non *d*-critical graph *G*, thus $\theta(G) \geq 1$. Next consider the graph H in Figure 2, which contains graph *G* as an induced subgraph, whose vertex set *V*(*H*) = *V*(*G*) ∪ {*v*} and the edge set *E*(*H*) = *E*(*G*) ∪ {u_1v, vu_5}. The partitions

$$D = \{ D_1 = \{u_1, u_4\}, D_2 = \{u_2, u_5\}, D_3 = \{u_3, v\}\}$$

are the *d*-partitions of H and thus $d(H) \geq 3$ with $d(H) \leq \delta(H) + 1$. Hence $d(H) = 3$ $= \delta(H) + 1$, i.e., H is domatically full. Also it can be seen from Figure 2 that every edge of H is incident with vertex of minimum degree 2. By Theorem 2.4, H is *d*-critical and it contains G as an induced subgraph. Hence $\theta(G) = 1$.

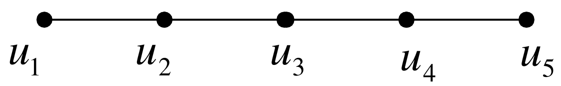

Figure 1. Non *d*-critical graph *G*.

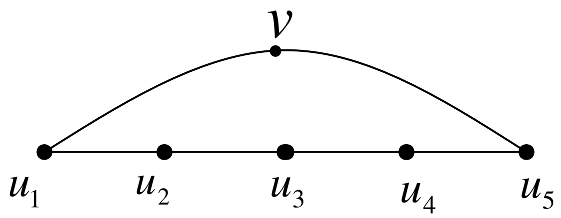

Figure 2. *d*-critical graph *H* (containing *G*).

3 Definitions

Definition 3.1. A graph *G* is called *indominable*, if its vertex set can be partitioned into independent dominating sets.

Definition 3.2. A dominating set *d* of a Graph *G* is called an *independent dominating set* of *G* if *d* is independent in *G*.

4 Main results

Theorem 4.1. Let $G = P_m \times P_n$, $m = 2$ and $n = 3$. Then, $\theta(G) = 2$.

Proof. Let *H* be a *d*-critical graph containing *G* has an induced subgraph with $d(H) = k \geq 2$. Then, by Theorem 2.1 the vertex set $V(H)$ can be partitioned into dominating sets $D_1, D_2, \ldots, D_k$ such that D_i is independent and $\langle D_i \cup D_j \rangle$ is union of stars for $i \neq j$ and

$i, j \in \{1, 2, \ldots, k\}$. Define $D_i^1 = D_i \cap V(G)$ for $i = 1, 2, \ldots, k$. Then, D_i^1, $i = 1, 2, \ldots, k$, are independent and $\langle D_i^1 \cup D_j^1 \rangle$ is a union of either stars or isolated vertices or both or empty graphs.

Therefore, the vertices of G are distributed in D_i for $i = 1, 2, \ldots, k$. Thus each D_i will contain at least one vertex other than the vertices of G. Thus $p(H) \geq 6 + k \geq 8$ since $k \geq 2$. Therefore, $\theta(G) \geq 2$.

Label the vertices of the grid G as $u_1, u_2, u_3, u_4, u_5, u_6$ as shown in Figure 3.

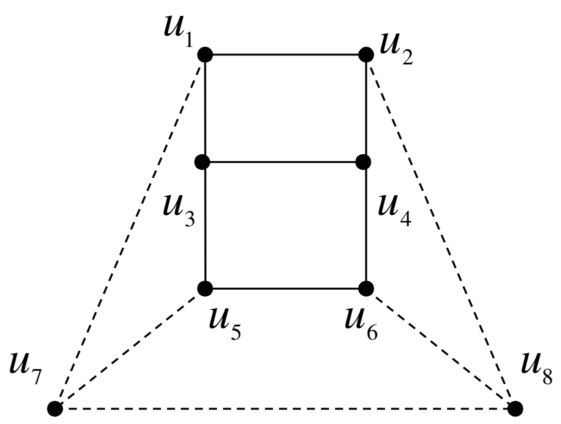

Figure 3. Graph H containing the grid graph G as an induced subgraph.

Let H be a super graph containing the grid G as an induced subgraph, which is obtained from G by adding two new vertices u_7 and u_8 such that vertex u_7 is adjacent to the vertices $\{u_5, u_1, u_8\}$ and vertex u_8 is adjacent to the vertices $\{u_2, u_6, u_7\}$. Then, $D_1 = \{u_3, u_8\}$, $D_2 = \{u_4, u_7\}$, $D_3 = \{u_1, u_6\}$, $D_4 = \{u_2, u_4\}$ are the domatic partitions of H. Thus $d(H) \geq 2$ but $d(H) \leq \delta(H) + 1 = 4$. Further H is d-critical since every edge in H is incident with $\delta(H) = 3$ and domatically full. Therefore, $\theta(G) \leq 2$.

Hence, $\theta(G) = 2$. □

Theorem 4.2. Let $G = P_m \times P_n$, $m = 2$ and $n = 4$. Then, $\theta(G) = 4$.

Proof. Let H be a d-critical graph containing G has an induced subgraph with $d(H) = k \geq 4$. Then, by Theorem 2.1 the vertex set $V(H)$ can be partitioned into dominating sets $D_1, D_2, \ldots, D_k$, such that D_i is independent and $\langle D_i \cup D_j \rangle$ is union of stars for $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$. Define $D_i^1 = D_i \cap V(G)$ for $i = 1, 2, \ldots, k$. Then, D_i^1, $i = 1, 2, \ldots, k$, are independent and $\langle D_i^1 \cup D_j^1 \rangle$ is a union of either stars or isolated vertices or both or empty graphs.

Therefore, the vertices of G are distributed in D_i for i = 1, 2, …, k. Thus each D_i will contain at least one vertex other than the vertices of G. Thus $p(H) \geq 8 + k \geq 12$ since $k \geq 4$. Therefore, $\theta(G) \geq 4$.

Label the vertices of the grid G as $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ as shown in Figure 4.

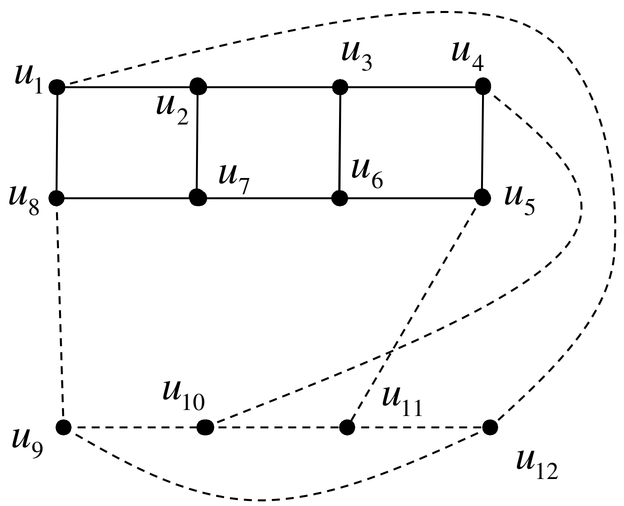

Figure 4. Graph H containing the grid graph G as an induced subgraph.

Let H be a super graph containing the grid G as an induced subgraph which is obtained from G by adding four new vertices $u_9, u_{10}, u_{11}, u_{12}$ such that vertex u_9 is adjacent to the vertices {u_8, u_{10}, u_{12}}, vertex u_{10} is adjacent to the vertices {u_9, u_{11}, u_4}, vertex u_{11} is adjacent to the vertices {u_5, u_{10}, u_{12}} and vertex u_{12} is adjacent to the vertices {u_1, u_9, u_{11}}. Then, $D_1 = \{u_7, u_4, u_{12}\}$, $D_2 = \{u_6, u_1, u_{10}\}$, $D_3 = \{u_2, u_5, u_9\}$, $D_4 = \{u_3, u_8, u_{11}\}$ are the domatic partitions of H. Thus $d(H) \geq 4$ but $d(H) \leq \delta(H) + 1 = 4$. Further H is d-critical since every edge in H is incident with $\delta(H) = 3$ and domatically full. Therefore, $\theta(G) \leq 4$.

Hence, $\theta(G) = 4$. □

Theorem 4.3. Let $G = P_m \times P_n$, $m = n = 3$. Then, $\theta(G) = 6$.

Proof. Let H be a d-critical graph containing G has an induced subgraph with $d(H) = k \geq 6$. Then, by Theorem 2.1 the vertex set $V(H)$ can be partitioned into dominating sets $D_1, D_2, \ldots, D_k$ such that D_i is independent and $\langle D_i \cup D_j \rangle$ is union of stars for $i \neq j$ and $i, j \in$ {1, 2, …, k}. Define $D_i^1 = D_i \cap V(G)$ for i = 1, 2, …, k. Then, D_i^1 i = 1, 2, …, k, are independent and $\langle D_i^1 \cup D_j^1 \rangle$ is a union of either stars or isolated vertices or both or empty graphs.

Therefore, the vertices of *G* are distributed in D_i for *i* = 1, 2, …, *k*. Thus each D_i will contain at least one vertex other than the vertices of *G*. Thus $p(H) \geq 9 + k \geq 15$ since $k \geq 6$. Therefore, $\theta(G) \geq 6$.

Label the vertices of the grid G as $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9$ as shown in Figure 5. Let *H* be a super graph containing the grid G as an induced subgraph.

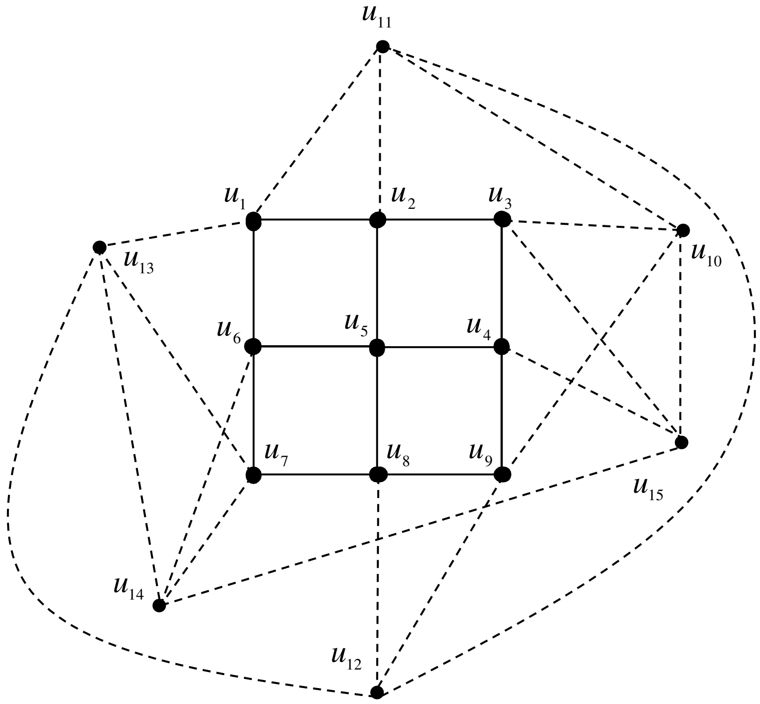

Figure 5. Graph *H* containing the grid graph $G = P_3 \times P_3$ as an induced subgraph.

Then, *H* is obtained by adding six new vertices $u_{10}, u_{11}, u_{12}, u_{13}, u_{14}$ and u_{15} such that vertex u_{10} is adjacent to the vertices $\{u_3, u_{15}, u_9, u_{11}\}$, vertex u_{11} is adjacent to the vertices $\{u_1, u_2, u_{10}, u_{12}\}$, vertex u_{12} is adjacent to the vertices $\{u_8, u_9, u_{11}, u_{13}\}$, vertex u_{13} is adjacent to the vertices $\{u_1, u_7, u_{12}, u_{14}\}$, vertex u_{14} is adjacent to the vertices $\{u_6, u_7, u_{13}, u_{15}\}$ and vertex u_{15} is adjacent to the vertices $\{u_3, u_4, u_{10}, u_{14}\}$. Then, $D_1 = \{u_1, u_8, u_{15}\}$, $D_2 = \{u_3, u_6, u_{12}\}$, $D_3 = \{u_5, u_{10}, u_{13}\}$, $D_4 = \{u_2, u_9, u_{14}\}$, $D_5 = \{u_4, u_7, u_{11}\}$ are the domatic partitions of *H*. Thus, $d(H) \geq 5$ but $d(H) \leq \delta(H) + 1 = 5$. Further *H* is *d*-critical since every edge in *H* is incident with $\delta(H) = 4$ and domatically full. Therefore, $\theta(G) \leq 6$.

Hence $\theta(G) = 6$. □

Finding the embedding index of the graph is a difficult task especially when the cardinality of the vertices in a graph *G* is large. Constructing the graph *H* such that *G* is an induced subgraph of *H* and *H* being *d*-critical is a tedious task. Hence we must use an alternate method to find the embedding index of *G* without actually drawing the exact graph *H*. This can be done using Theorems 2.2 and 2.3.

5 Non *d*-critical grid graphs

5.1 Let $G = P_m \times P_n$, $m = 2$ and $n = 3$. Then, the different possibilities of adding new edges to G in order to convert G into a three regular graph are shown below in Figure 6.

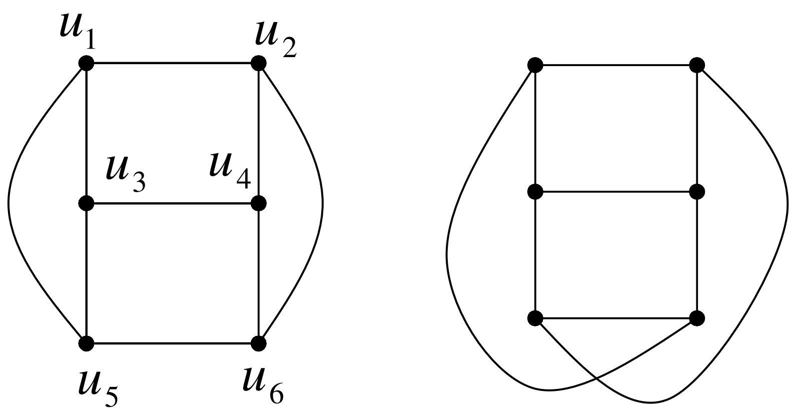

Figure 6. Three regular graph, obtained from the graph $P_2 \times P_3$.

From the above figure, $D_1 = \{u_1, u_6\}$, $D_2 = \{u_2, u_5\}$, $D_3 = \{u_3, u_4\}$ are the domatic partitions of *G*, Therefore, $d(G) = 3$, but $d(G - u_2u_6) = d(G)$. Therefore, *G* is non *d*-critical.

5.2 Let $G = P_m \times P_n$, $m = 2$ and $n = 4$. Then, the different possibilities of adding new edges to G in order to convert G into a three regular graph are shown below in Figure 7.

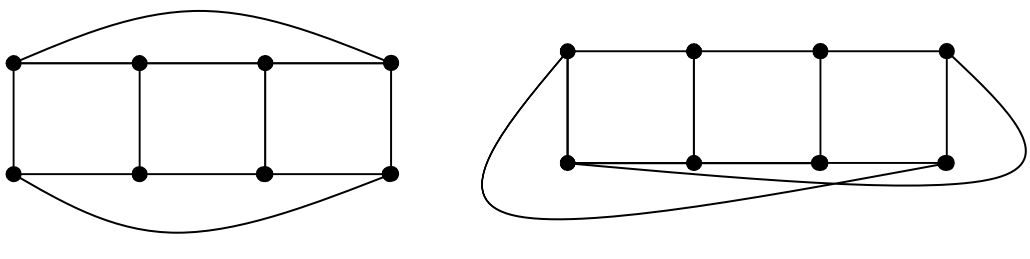

Figure 7. Three regular graph, obtained from the graph $P_2 \times P_4$.

Similar to the above observation it can be shown that $d(G) = d(G–x)$, $x \in E(G)$. Therefore, *G* is non *d*-critical.

5.3 Let $G = P_m \times P_n$, $m = n = 3$. Then, a few of the different possibilities of adding new edges to *G* in order to convert *G* into a four regular graph are shown in Figure 8.

Similar to the above observation it can be shown that $d(G) = d(G–x)$, $x \in E(G)$. Therefore, *G* is non *d*-critical.

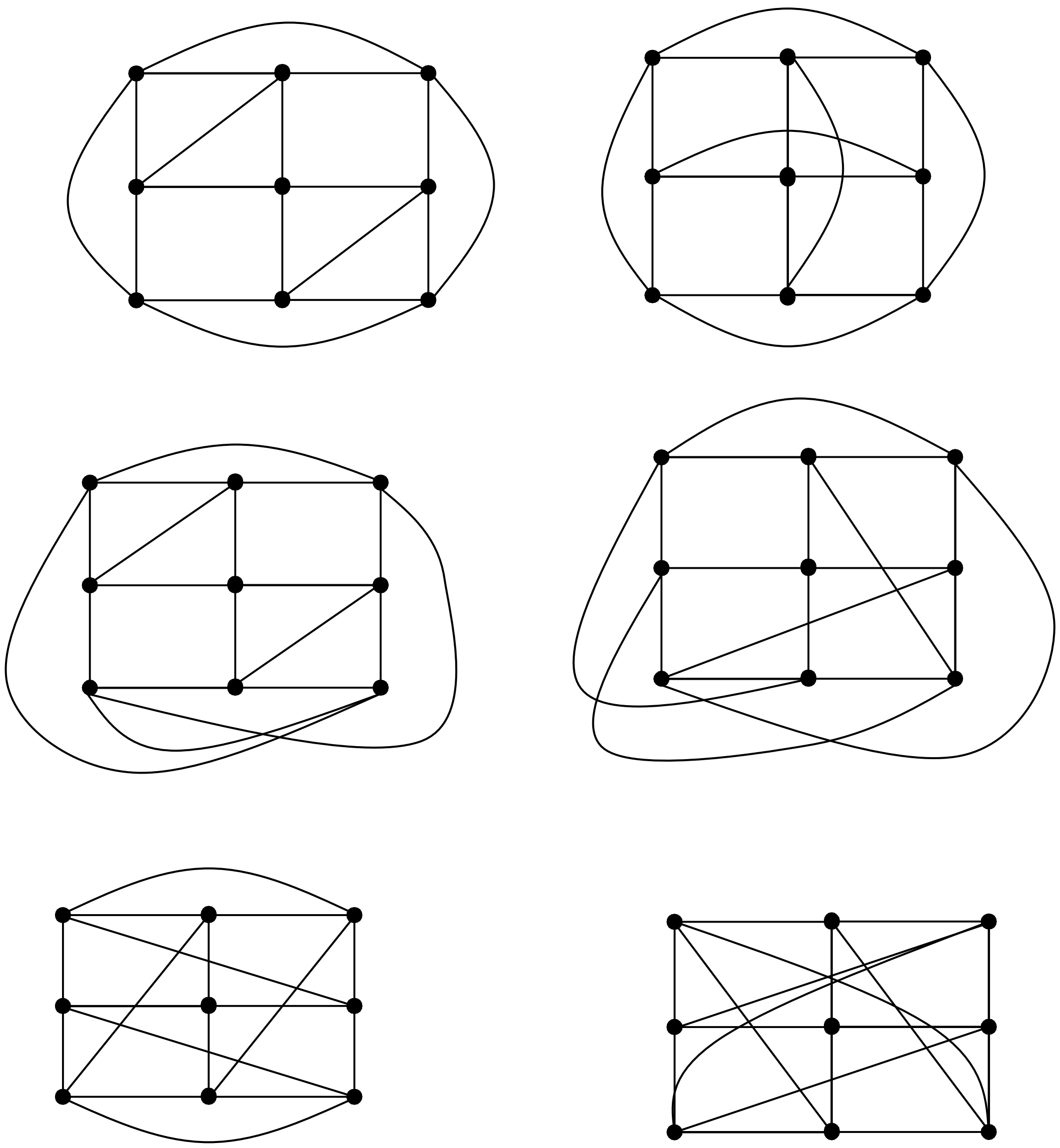

Figure 8. Four regular graph, obtained from the graph $P_3 \times P_3$.

The above result can be generalized.

Lemma 1. Let $G = P_m \times P_n$, $m = 3$ and $n = 3k$, $k \in N$. Then, constructing a regular graph by adding new edges to *G* does not convert *G* to *d*-critical.

Proof. Let $G(V, E) = P_3 \times P_{3k}$, $|V| = 9k$, $|E| = 2\times 3k + (3k-1)\times 3$, $|E| = 15k - 3$. Label the grid *G* as $u_1, u_2, ..., u_{3k}$, $v_1, v_2, ..., v_{3k}$ and $w_1, w_2, ..., w_{3k}$ as shown in Figure 9.

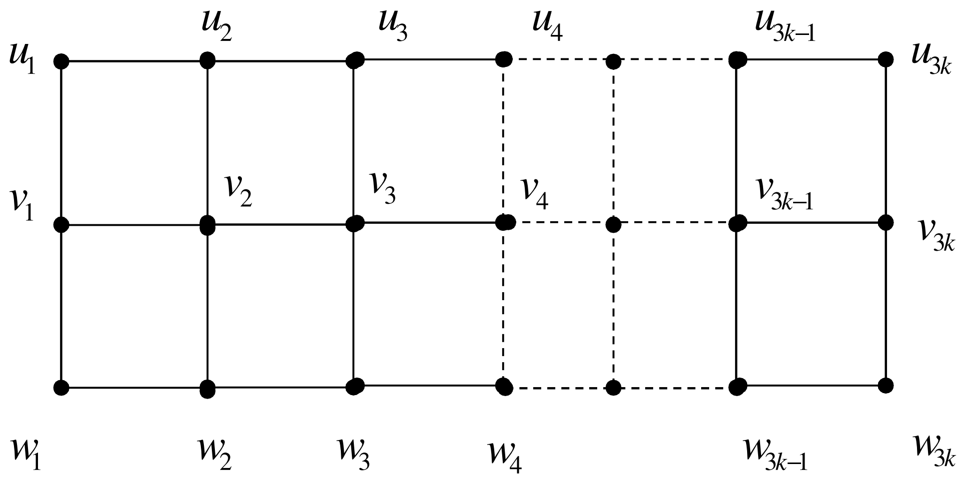

Figure 9. Grid graph, $G = P_3 \times P_{3k}$.

In G, vertices $v_2, v_3, ..., v_{3k-1}$ are of degree four, $v_1, v_{3k}, u_2, u_3, ..., u_{3k-1}, w_2, w_3, ..., w_{3k-1}$ are of degree three and u_1, u_{3k}, w_1, w_{3k} are of degree two. Maximum degree of G is four. Therefore, by adding new edges to G, the graph G can be made a four regular graph. Let the domatic number $d(G) = d$. Then, the vertex set $V(G)$ of G is the union of d pairwise disjoint sets $V_1, V_2, V_3, ..., V_d$ with the property that for any two distinct integers i, j ; $1 \le i, j \le d$, the subgraph G_{ij} of G induced by the set $V_i \cup V_j$ may be a bipartite graph on the sets V_i, V_j, all of whose connected components are not stars. This contradicts Theorem 2.1.

Hence the four regular graph G thus constructed is not d-critical.

The same result can be shown for other regular graphs which can be constructed for G. Hence the proof. □

6 Upper bound of grid graphs

Theorem 6.1. Let $G = P_m \times P_n$, $m = n = 3$. Then, $\theta(G) \le 6$.

Proof. Graph G is non d-critical and non-regular. To apply Theorem 2.2, graph G has to be regular. Since the maximum degree of G is four, G can be converted into a four or higher degree regular graph.

Case (i). We first convert G into a four regular graph. Label the vertices of G as $u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3$ as shown in Figure 10. Two new edges must be incident to vertices u_1, u_3, w_1, w_3 and one new edge must be incident to u_2, w_2, v_1, v_3 in order to convert G into a four regular graph. Hence minimum of three new vertices must be added to G in order to convert G into a four regular graph. Let H be a super graph containing $G = P_3 \times P_3$ as an induced subgraph. The graph $H(V_1, E_1)$ is obtained by adding three new vertices to G. Therefore, $|V_1| = 9 + 3 = 12$ and $|E_1| = 12 + 12 = 24$. From Theorem 2.2, H is d-critical if d/n.

$d = 5$ as H is four regular graph and domatically full. Further H is non d-critical as $d = 5$ does not divide $n = 12$. So minimum of three more new vertices must be added, so that d divides n.

Therefore, $\theta(P_3 \times P_3) = 6$.

Since we assume that H is regular θ so obtained is the upper bound for embedding index of G. If there exists a graph H being non-regular and d-critical containing G as an induced subgraph Then, $\theta(P_3 \times P_3) \leq 6$.

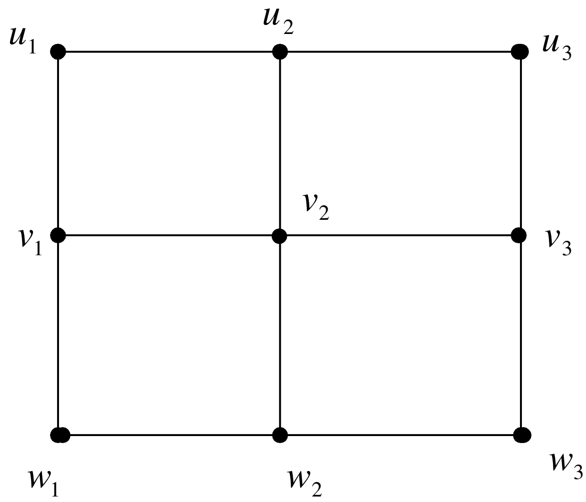

Figure 10. Grid graph $P_3 \times P_3$.

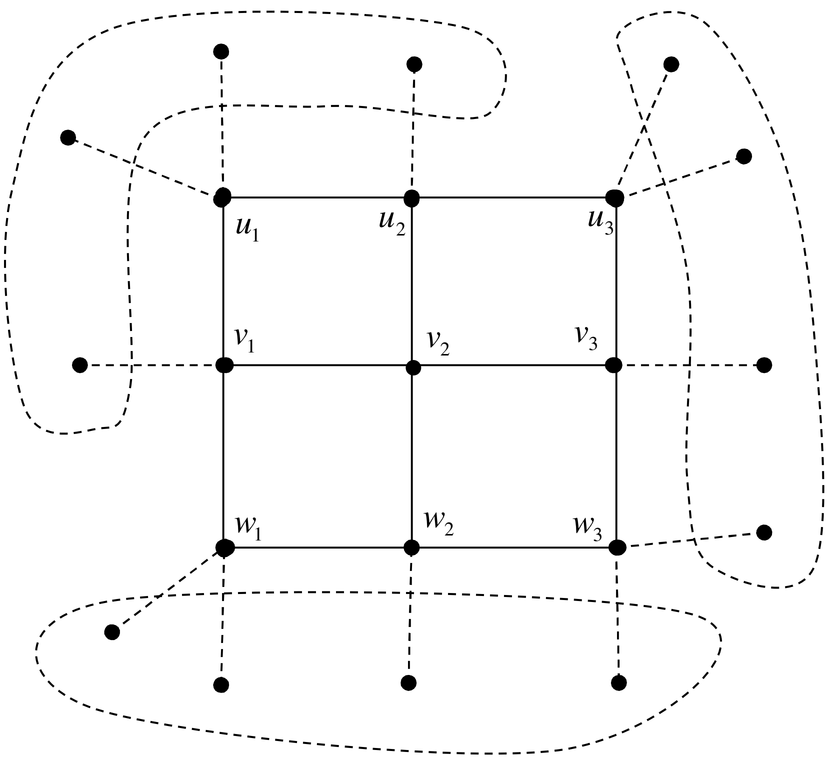

Figure 11. A four regular graph H containing the grid graph G as an induced subgraph.

Case (ii). If G is converted into a five regular graph, using the above explanation we have d=6, minimum of five new vertices must be added to G in order to convert G into a five regular graph (see Figure 12). Further $d = 6$ does not divide $n = 14$, so a minimum of four more new vertices must be added so that d divides n.

Hence $\theta(P_3 \times P_3) \leq 9$.

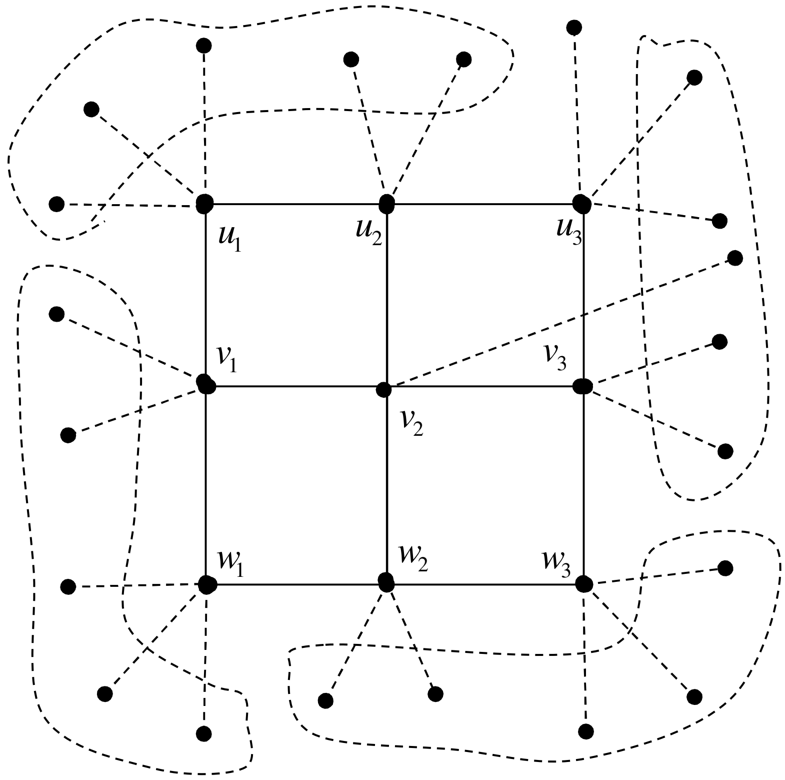

Figure 12. A five regular graph H containing the grid graph G as an induced subgraph.

The embedding index of a non d-critical graph G is the smallest order of d-critical graph H containing G as an induced subgraph. Therefore, we can conclude that H has to be constructed with minimum regular.

Hence the proof. □

Theorem 6.2. Let $G = P_m \times P_n$, $m = 3$ and $n = 3k$. Then, $\theta(G) \leq a_k$, where a_k is the recurrence relation given by $a_k = \begin{cases} a_{k-1} + 6 & k = 10s - 2, s \in N \\ a_{k-1} + 1, a_0 = 5 & \text{otherwise} \end{cases}$, $k \in N$.

Proof. For $k = 1$, the result is proved in Theorem 4.3.

Let $k > 1$. From Theorem 6.1, we obtain $a_0 = 5$, let $G(V,E) = P_3 \times P_{3k}$, $|V| = 9k$, $|E| = 2\times 3k + (3k-1)\times 3$, $|E| = 15k - 3$. Label the grid G as $u_1, u_2, ..., u_{3k}$, $v_1, v_2, ..., v_{3k}$ and $w_1, w_2, ..., w_{3k}$. In G, vertices $v_2, v_3, ..., v_{3k-1}$ are of degree four, $v_1, v_{3k}, u_2, u_3, ..., u_{3k-1}$, $w_2, w_3, ..., w_{3k-1}$ are of degree three and u_1, u_{3k}, w_1, w_{3k} are of degree two. To convert graph G

into a four regular graph, $2(3k)+6$ new edges must be added to *G*.

Hence minimum number of new vertices required to be added in *G* to convert it into a four regular graph is given by $\lceil 6(k+1)/4 \rceil$. Let *H* be a super graph containing $P_3 \times P_{3k}$ as an induced subgraph. The graph *H* is obtained by adding new $\lceil 6(k+1)/4 \rceil$ vertices to *G*. From Theorem 2.2, *H* is *d*-critical, if *d/n*. i.e., if 5 divides $9k+\lceil 6(k+1)/4 \rceil$ Then, *H* is *d*-critical. Hence the recurrence relation for $\theta(G)$ function is given by $a_{k-1}+6$ for $k=10s-2, s\in N$ and $a_{k-1}+1$ otherwise, $k \in N$. Furthermore *H* is *d*-critical since every edge in *H* is incident with $\delta(H)=4$ and it is domatically full.

Hence the proof. □

Theorem 6.3. Let $G=P_m \times P_n$, $m=3$ and $n=3k+1$. Then, $\theta(G)\le a_k$, where a_k is the recurrence relation given by $a_k = \begin{Bmatrix} a_{k-1}+6 & k=10s+1, s\in N \\ a_{k-1}+1, a_0=7 & \text{otherwise} \end{Bmatrix}$, $k \in N$.

Proof. Similar to the proof of Theorem 6.1, we can show that $a_0=7$, let $G(V,E)=P_3\times P_{3k+1}$, $|V|=9k+3$, $|E|=2\times(3k+1)+(3k)\times 3$, $|E|=15k+2$. Label the grid *G* as $u_1,u_2,...,u_{3k+1}$, $v_1,v_2,...,v_{3k+1}$, $w_1,w_2,...,w_{3k+1}$. In *G* vertices $v_2,v_3,...,v_{3k}$ are of degree four, v_1,v_{3k+1}, $u_2,u_3,...,u_{3k}$ and $w_2,w_3,...,w_{3k}$ are of degree three and $u_1,u_{3k+1},w_1,w_{3k+1}$ are of degree two. To convert graph *G* into a four regular graph, $2(3k+1)+6$ new edges must be added to *G*.

Hence, minimum number of new vertices required to be added in *G* to convert it into a four regular graph is given by $\lceil (6k+8)/4 \rceil$. Let *H* be a super graph containing *G* as an induced subgraph. The graph *H* is obtained by adding $\lceil (6k+8)/4 \rceil$ new vertices to *G*. From Theorem 2.2, if *d/n* Then, *H* is *d*-critical, i.e., if 5 divide $(9k+3)+\lceil (6k+8)/4 \rceil$ Then, *H* is *d*-critical. Hence the recurrence relation for $\theta(G)$ function is given by $a_{k-1}+6$ for $k=10s+1$, $s\in N$ and $a_{k-1}+1$ otherwise, $k\in N$. Furthermore, *H* is *d*-critical since every edge in *H* is incident with $\delta(H)=4$ and it is domatically full.

Hence the proof. □

Theorem 6.4. Let $G=P_m \times P_n$, $m=3$ and $n=3k+2$. Then, $\theta(G)\le a_k$, where a_k is the recurrence relation given by $a_k = \begin{Bmatrix} a_{k-1}+6 & k=10s-6, s\in N \\ a_{k-1}+1, a_0=4 & \text{otherwise} \end{Bmatrix}$, $k \in N$.

Proof. Similar to the proof of Theorem 6.1 we can show that $a_0=4$, let $G(V,E)=P_3\times P_{3k+2}$, $|V|=9k+6$, $|E|=2\times(3k+2)+(3k+1)\times 3$, $|E|=15k+7$. Label the grid *G* as $u_1,u_2,...,u_{3k+2}$, $v_1,v_2,...,v_{3k+2}$, $w_1,w_2,...,w_{3k+2}$. In *G* vertices $v_2,v_3,...,v_{3k+1}$ are of degree four, v_1,v_{3k+2}, $u_2,u_3,...,u_{3k+1}$ and $w_2,w_3,...,w_{3k+1}$ are of degree three and $u_1,u_{3k+2},w_1,w_{3k+2}$ are of degree two. To convert graph *G* into a four regular graph, $2(3k+2)+6$ new edges must be added to *G*.

Hence, minimum number of new vertices required to be added in *G* to convert it into a four regular graph is given by $\lceil (6k+10)/4 \rceil$. Let *H* be a super graph containing *G* as an induced subgraph. The graph *H* is obtained by adding $\lceil (6k+10)/4 \rceil$ new vertices to *G*. From Theorem 2.2, if *d/n* Then, *H* is *d*-critical, i.e., if 5 divide $(9k+6)+\lceil (6k+10)/4 \rceil$ Then, *H* is *d*-critical. Hence the recurrence relation for $\theta(G)$ function is given by $a_{k-1}+6$ for $k=10s-6$, $s \in N$ and $a_{k-1}+1$ otherwise, $k \in N$. Furthermore, *H* is *d*-critical since every edge in *H* is incident with $\delta(H)=4$ and it is domatically full.

Hence the proof. □

Theorem 6.5. Let $G = P_m \times P_n$, $m=2$ and $n=2k$. Then, $\theta(G) \le 4$, $k>1$, $k \in N$.

Proof. Let $G(V,E) = P_2 \times P_{2k}$, $|V| = 4k$, $|E| = 2k+(2k-1)\times 2$, $|E| = 6k-2$. Label the grid *G* as $u_1, u_2, ..., u_{2k}$, $v_1, v_2, ..., v_{2k}$. In *G* vertices $u_2, u_3, ..., u_{2k-1}$, $v_2, v_3, ..., v_{2k-1}$ are of degree three, v_1, v_{2k}, u_2, u_{2k} are of degree two. To convert graph *G* into a three regular graph, four new edges must be added to *G*.

Hence minimum number of new vertices required to be added in *G* to convert it into a three regular graph is two. Let *H* be a super graph containing *G* as an induced subgraph. The graph *H* is obtained by adding two new vertices to *G*. From Theorem 2.2, if *d/n* Then, *H* is *d*-critical, i.e., if 4 divide $4k+\lceil 4/3 \rceil$ Then, *H* is *d*-critical. Hence two more new vertices must be added to *H* such that 4 divide 4*k*+4.

Hence the proof. □

Theorem 6.6. Let $G = P_m \times P_n$, $m=2$ and $n=2k+1$. Then, $\theta(G) \le 2$, $k \in N$.

Proof. Let $G(V,E) = P_2 \times P_{2k+1}$, $|V| = 4k+2$, $|E| = 2k+1+(2k)\times 2$, $|E| = 6k+1$. Label the grid *G* as $u_1, u_2, ..., u_{2k+1}, v_1, v_2, ..., v_{2k+1}$. In *G* vertices $u_2, u_3, ..., u_{2k}$, $v_2, v_3, ..., v_{2k}$ are of degree three, $v_1, v_{2k+1}, u_2, u_{2k+1}$ are of degree two. To convert graph *G* into a three regular graph, four new edges must be added to *G*.

Hence minimum number of new vertices required to be added in *G* to convert it into a three regular graph is two. Let *H* be a super graph containing *G* as an induced subgraph. The graph *H* is obtained by adding two new vertices to *G*. From theorem 2.2, if *d/n* Then, *H* is *d*-critical, i.e., if 4 divides $4k+2+\lceil 4/3 \rceil$ Then, *H* is *d*-critical.

Hence the proof. □

The embedding index of the graph $P_m \times P_n$ for few values of *m* and *n* are discussed in this section. One can find the embedding index of any grid $P_m \times P_n$ using proof similar to the above theorems. The upper bound so obtained for any grid $P_m \times P_n$ will be sharp as there is no graph *H* being irregular, *d*-critical and containing grid $P_m \times P_n$ as an induced subgraph.

The exact value of the embedding index for a tadpole graph $T(3,n)$ is already proved in [12]. Here we find the upper bound for the same using Theorems 2.2 and 2.3. We show that the bounds are not sharp as there exist an irregular *d*-critical graph *H* containing $T(3,n)$ as an induced subgraph. Hence, there is a clear question for which kind of graphs the bounds are sharp. This problem is open.

Theorem 6.7. Let $G = T(3,3n)$. Then, $\theta(G) \le n+1, n \in N$.

Proof. Let $G(V,E) = T(3,3n)$, $|V| = 3n+3$, $|E| = 3n+3$. Label the vertices of the cycle as u_1, u_2, u_3 and vertices of the path as $v_1, v_2, v_3, ..., v_{3n}$. In *G*, u_3 is of degree three, vertices $v_1, v_2, v_3, ..., v_{3n-1}$ are of degree two, vertex v_{3n} is of degree one. To convert graph *G* into a three regular graph, $3n+3$ new edges must be added to *G*.

Hence, minimum number of new vertices required to be added in *G* to convert it into a three regular graph is given by $\lceil (3n+3)/3 \rceil = n+1$. Let *H* be a super graph containing *G* as an induced subgraph. The graph *H* is obtained by adding $n+1$ new vertices to *G*. From Theorem 2.2, if *d/n* Then, *H* is *d*-critical, i.e., if 4 divides $3n+3+n+1 = 4n+4$ Then, *H* is *d*-critical.

Hence the proof. □

Theorem 6.8. Let $G = T(3,3n-1)$. Then, $\theta(G)) \le n+2, n \in N$.

Proof. Let $G(V,E) = T(3,3n-1)$, $|V| = 3n+2$, $|E| = 3n+2$. Label the vertices of the cycle as u_1, u_2, u_3 and vertices of the path as $v_1, v_2, v_3, ..., v_{3n-1}$. In *G*, u_3 is of degree three, vertices $v_1, v_2, v_3, ..., v_{3n-2}$ are of degree two, vertex v_{3n-1} is of degree one. To convert graph *G* into a three regular graph, $3n+2$ new edges must be added to *G*.

Hence minimum number of new vertices required to be added in *G* to convert it into a three regular graph is given by $\lceil (3n+2)/3 \rceil = n+1$. Let *H* be a super graph containing *G* as an induced subgraph. The graph *H* is obtained by adding $n+1$ new vertex to *G*. From theorem 2.2, if *d/n* Then, *H* is *d*-critical, i.e., if 4 divides $3n+2+n+1 = 4n+3$. Hence one more vertex must be added to *G* so that 4 divide $4n+4$, Then, *H* is *d*-critical.

Hence the proof. □

Theorem 6.9. Let $G = T(3,3n-2)$. Then, $\theta(G) \le n+3, n \in N$.

Proof. Let $G(V,E) = T(3,3n-2)$, $|V| = 3n+1$, $|E| = 3n+1$. Label the vertices of the cycle as u_1, u_2, u_3 and vertices of the path as $v_1, v_2, v_3, ..., v_{3n-2}$. In *G*, u_3 is of degree three, vertices $v_1, v_2, v_3, ..., v_{3n-3}$ are of degree two, vertex v_{3n-2} is of degree one. To convert graph *G* into a three regular graph, $3n+1$ new edges must be added to *G*.

Hence minimum number of new vertices required to be added in *G* to convert it into a three regular graph is given by $\lceil (3n+1)/3 \rceil = n+1$. Let *H* be a super graph containing *G* as an induced subgraph. The graph *H* is obtained by adding $n+1$ new vertex to *G*. From

Theorem 2.2, if *d/n* Then, *H* is *d*-critical, i.e., if 4 divides $3n+1+n+1= 4n+2$. Hence two new vertices must be added to *G* so that 4 divide $4n+4$, Then, *H* is *d*-critical.

Hence the proof. □

7 Upper bound for the embedding index in the generalized grid graph

Theorem 7.1. For any grid

$$G = P_m \times P_n,$$

$$\theta(P_m \times P_n) \le \begin{cases} \left\lceil \frac{m+n}{2} \right\rceil + (5-L), & L \neq 0 \\ \left\lceil \frac{m+n}{2} \right\rceil, & L = 0 \end{cases},$$

where L is such that $k \equiv L \bmod 5$ and $k = \left\lceil \frac{2mn+m+n}{2} \right\rceil$, $m,n \in N$, $m,n > 2$.

Proof. For any $G = P_m \times P_n$, $|V| = m \times n$, $|E| = 2mn - m - n$. Let the vertices of G be $u_{11}, u_{12}, \ldots, u_{1n}$, $u_{21}, u_{22}, \ldots, u_{2n}$, $\ldots$, $u_{m1}, u_{m2}, \ldots, u_{mn}$. To convert G into a four regular graph, two new edges must be incident at $u_{11}, u_{1n}, u_{m1}, u_{mn}$ and one edge must be incident at $u_{12}, u_{13}, \ldots, u_{1,n-1}$, $u_{21}, u_{31}, \ldots, u_{m-1,1}$, $u_{2n}, u_{3,n}, \ldots, u_{m-1,n}$, and $u_{m2}, u_{m3}, \ldots, u_{m,n-1}$.

Hence, $2(m + n)$ new edges must be added to $G = P_m \times P_n$ in order to convert G into a four regular graph and a minimum of $\left\lceil \frac{2(m+n)}{4} \right\rceil$ new vertices must be added to *G*. Let *H* be a super graph containing $G = P_m \times P_n$ as an induced subgraph. The graph $H(V_1, E_1)$ is obtained by adding $\left\lceil \frac{(m+n)}{2} \right\rceil$ new vertices to *G*. $|V_1| = mn + \left\lceil \frac{m+n}{2} \right\rceil$, $|E_1| = 2mn + m + n$. From Theorem 2.2, *H* is *d*-critical if *d/n*. $d = 5$ as *H* is four regular and domatically full. Therefore, $5 / mn + \left\lceil \frac{m+n}{2} \right\rceil \Rightarrow 5 / \left\lceil \frac{2mn+m+n}{2} \right\rceil$. Five cases arise in this division i.e., the division can be $5r$, $5r + 1$, $5r + 2$, $5r + 3$ or $5r + 4$. Hence let $k = \left\lceil \frac{2mn+m+n}{2} \right\rceil$. In order to make *k* divisible by 5, let $k \equiv L \bmod 5$. Hence $(5-L)$ more new vertices must be added so that 5 divide *k*.

Hence the proof. □

8 Conclusion

In this paper, an alternative method to find the embedding index of a graph *G* is given. This method calculates the embedding index without actually drawing the graph *H*.

The embedding index of the grid graphs $P_2 \times P_3$, $P_2 \times P_4$ and $P_3 \times P_3$ was determined.

We have presented an idea of constructing the graph H and have obtained the upper bound of the embedding index for generalized grid graph $G = P_m \times P_n$. The upper bound so obtained is sharp.

References

[1] Zelinka, B. (1980) Domatically critical graphs, *Czechoslovak Math. L.* 30(3), 486–489.

[2] Rall, D. F. (1990) Domatically critical and domatically full graphs, *Disc. Math.* 86, 81–87.

[3] Cockayne, E. L., & Hedetniemi, S. T. (1976) Disjoint independent dominating sets in graphs, *Disc. Math.* 15, 213–222.

[4] Cockayne, E. L., & Hedetniemi, S. T. (1977) Towards a Theory of domination in graphs, *Networks*, 7, 247–261.

[5] Harary, F. (1972) *Graph Theory*, Addison Wesley, Reading Mass.

[6] Dirac, G. A. (1952) A property of 4-chromatic graphs and some remarks on critical graphs, *Journal of London Mathematical Society*, 27, 85–92.

[7] Walikar, H. B., & Deshpande, A.P. (1994) Domatically critical graphs, *Proc. of National Seminar on Recent Developments in Mathematics*, Karnatak University, Dharwar, 27–30.

[8] Walikar, H. B., & Savitha Basapur (1996) Deficiency of non *d*-critical graphs, *Proc. of National Workshop on Graph Theory and Its Applications*, Manomaniam Sundaranar University, Tirunelveli, Feb 21-27, 1996, 235–242.

[9] Walikar, H. B. (1996) Domatically Critical and Co-critical Graphs, *Proc. of National Workshop on Graph Theory and Its Applications*, Manomaniam Sundaranar University, Tirunelveli, Feb 21-27, 1996, 223–234.

[10] Walikar, H. B., & Acharya, B.D. (1979) Domination critical graphs, *National academy of sciences*, 2, 70–72.

[11] Sudershan Reddy, L. (2006) *The theory of domination in graphs & related topics*, *PhD Thesis*, Karnatak University.

[12] Kamal Kumar, M., & Murali, R. (2015) Embedding index in some class of graphs, *Notes on Number Theory and Discrete Mathematics*, 21(2), 80–88.

[13] Arumugam, S., & Raja Chandrasekhar, K. (2012) Minimal Dominating sets in Maximum domatic partitions, *Australian Journal of Combinatorics*, 52, 281–292.