A note on Dedekind’s arithmetical function

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Abstract: We point out that an inequality published recently in [1] is a particular case of a general result from [4]. By another method, a refinement is offered, too. Related inequalities are also proved.

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1 Introduction

Let $\psi(n)$ be the Dedekind arithmetic function, defined by $\psi(1) = 1$, and $\psi(n) = n \prod_{i=1}^{r} \left(1 + \frac{1}{p_i}\right)$ for $n > 1$. Here $p_i$ denote the distinct prime divisors of $n$. In the recent paper [1], the following inequality is stated:

$$(\psi(ab))^k \geq \psi(a^k)\psi(b^k), \text{ for any } a, b \geq 1 \text{ and } k \geq 2 \quad (1)$$

We note that, (1) is a particular case of a result from our published paper (and arXiv Preprint) [4] (see Theorem 7 of [4]). Namely, for any arithmetical function $f$ satisfying

$$(f(ab))^k \geq f(a^k)f(b^k); a, b \geq 1, k \geq 2 \quad (2)$$

are called $k$-super multiplicative, and when (2) holds with reversed inequality (i.e. “$\leq$”), the $k$-submultiplicative functions.

In [4] it is proved that, if $f$ is a multiplicative function, with $f(1) = 1$, and for any integers $x, y \geq 0$, and $k \geq 1$, integer, $p$ prime, one has
\[(f(p^x y))^k \geq (f(p^{kx}) f(p^{ky})) \]

then (2) holds true, i.e., the function \(f\) is \(k\)-super multiplicative.

It is easy to see that, when \(f(n) = \psi(n)\), inequality (3) becomes
\[
(1 + \frac{1}{p})^k \geq (1 + \frac{1}{p})^2
\]
so clearly, (3) is true. We note that, for \(f(n) = \varphi(n)\) (i.e. Euler’s totient), in [4] (and arXiv Preprint) is proved the reverse inequality, but there are considered also many other particular cases.

### 2 Main results

The following refinement of (1) holds true:

**Theorem 1.** For any integers \(a, b \geq 1\) and \(k \geq 2\) one has
\[
(\psi(ab))^k \geq ((ab)^{k-2}) \cdot (\psi(ab))^2 \geq \psi(a^k) \psi(b^k)
\]

**Proof.** The following known properties of the function \(\psi(n)\) will be applied:
\[
\psi(n^k) = n^{k-1} \psi(n); \quad (5)
\]
\[
\psi(nm) \geq n \psi(m) \quad (6)
\]
for any integers \(n, k, m \geq 1\). For proofs, see e.g. [2], [3].

Now, the first relation of (4) may be rewritten as
\[
\left(\frac{\psi(ab)}{ab}\right)^k \geq \left(\frac{\psi(ab)}{ab}\right)^2 \quad (7)
\]
which is true, as \(k \geq 2\) and by \(\psi(n) \geq n\) one has \(\frac{\psi(ab)}{ab} \geq 1\). The second inequality of (4) may be rewritten as
\[
(\psi(ab))^2 \geq ab \psi(a) \psi(b) \quad (8)
\]
This is a consequence of relation (6) applied twice: \(\psi(ab) \geq a \psi(b)\) and \(\psi(ab) \geq b \psi(a)\). The proof of Theorem 1 is finished.

**Theorem 2.** For any integers \(n \geq k \geq 2\) and \(a, b \geq 1\) one has
\[
(\psi(ab))^k \geq (ab)^{n-k} \psi(a^n) \psi(b^n)
\]

**Proof.** Applying inequality (2) for \(a = x^n\) and \(b = y^n\) \((x, y \geq 1\) integers), and by taking into account of (5), after easy computations we get
\[
(\psi(xy))^k \geq (xy)^{n-k} \psi(x^n) \psi(y^n),
\]
which is in fact relation (9) with \(x\) and \(y\) in place of \(a\) and \(b\), respectively.
Remark. For \( n = k \), relation (9) implies inequality (1).

**Theorem 3.** For any integers \( a, b \geq 1 \) and \( k \geq 2 \) one has

\[
(ab)^{k-1} \frac{\psi(ab)}{\psi(a)\psi(b)} \leq (ab)^{k-1} \leq \frac{(\psi(ab))^k}{\psi(a)\psi(b)} \leq (\psi(a)\psi(b))^{k-1}
\]

(10)

**Proof.** Applying the property \( \psi(ab) \leq \psi(a)\psi(b) \) (see [2, 3]), and relations (2) and (5), we can write

\[
(ab)^{k-1} \psi(a)\psi(b) \leq (\psi(ab))^k \leq (\psi(a))^k(\psi(b))^k,
\]

which immediately gives the last two inequalities of (10). The first relation of (10) is in fact the above stated property (by reducing with \( (ab)^{k-1} \)). □

**References**


